ECAC-TR-65-1
[illegible]-TR-65-16

INTRODUCTION TO THE ANALYSIS OF INTERFERENCE TO THE PERFORMANCE OF COMMUNICATIONS SYSTEMS

MAY 1965

ELECTROMAGNETIC COMPATIBILITY ANALYSIS CENTER

A JOINT MILITARY FACILITY

ECAC

Prepared by R. Mayher, of the IIT Research Institute

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Technical Report

No. ECAC-TR-65-1

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Department of Defense
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Published by
Electromagnetic Compatibility Analysis Center
U.S. Navy Marine Engineering Laboratory
Annapolis, Maryland 21402

FOREWORD

The Electromagnetic Compatibility Analysis Center (ECAC) is a DOD facility, established to provide a data base, develop a rapid analysis capability and to provide analysis of electromagnetic compatibility projects of the military departments. The Center is under executive and administrative control of the Air Force and is located at the U.S. Navy Marine Engineering Laboratory, Annapolis, Maryland. The management and direction of the Center are provided by military and civil service personnel. The technical operations function is provided through an Air Force contract to the IIT Research Institute (IITRI).

This report was prepared as part of AF Project 649E under Contract AF 19(628)-5049 by the IIT Research Institute, ECAC, USNMEL, Annapolis, Maryland 21402.

Assistance in deriving the analysis contained in sections 8 through 12 of this report was provided by L.J. Greenstein, D. Fryberger, and B. Ebstein. Their work was accomplished as a part of the communication analysis task and was documented in an ECAC publication, see reference (1). In addition, Dr. R. Whiteman provided valuable assistance in technically reviewing the entire document.

Users of this report are invited to submit comments which would be useful in revising or adding to this material to the Director, ECAC, USNMEL, Annapolis, Maryland, Attention ACL.

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ABSTRACT

This is an introductory report discussing the analysis of the effects of unintentional interference on the performance of communications systems. The analysis includes interference signals of six off-tuned, non-design types of modulation and seven types of receivers. It contains a generally tractable problem approach and a general analysis of the cases considered and is introductory in the sense that only the most important of the large number of non-design modulation cases have been analyzed. The cases that have been analyzed consist of the desired to undesired signal types of AM to AM, FM to FM, SSB to SSB, AM to Pulse, FM to Pulse, SSB to Pulse, Pulse to Pulse, FSK to Pulse, AM to Noise plus Interference, FM to Noise plus Interference, PM to Noise plus Interference and both AM and FM Multiplex systems. The solutions obtained for the digital systems represent an almost complete performance evaluation in terms of the probability of false alarm and false dismissal. The solutions for analog and voice systems are partial performance solutions obtained as a function of the design parameters and signal-to-interference ratios. Although these results were derived for the case of unintentional interference the quantitative results also apply to similar situations involving intentional interference.

KEY WORDS

COMMUNICATION
INTERFERENCE
MODULATION
THEORY
MATHEMATICS
EFFECTIVENESS

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GLOSSARY OF SYMBOLS

A = Signal peak amplitude

A_{c1} = A function of R_s and the time moments of $m_s(t)$

AM = Amplitude modulation

$A_{S \text{ or } I}$ = Amplitude of the desired or the undesired signal

$A_{0,1,2}$ = Signal amplitude components

a_o^2 = Signal-to-noise ratio

$\overline{A_I^2(t)}$ = The normalized mean square interference modulation

B = Bandwidth

B_{IF} = IF half bandwidth

β_{IF} = IF bandwidth

$B_{S \text{ or } I}$ = Frequency modulation index of the desired or undesired signal

BW = Bandwidth

CW = Constant Wave

D_i = Peak frequency deviation of the ith channel

DSB-SC = Double sideband suppressed carrier

ΔV_{DC} = Pulse detected DC output term

$\Delta\omega$ = The radian frequency difference between the carrier of the desired and the undesired signal

Δf = The off-tuned frequency difference between the desired and the undesired carrier frequencies

δ = Pulse duty cycle

FDM = Frequency division multiplex

FSK = Frequency shift key

FM = Frequency modulation

$F(\omega)$ = A function of radian frequency

f_d = 3 db bandwidth

f_m = Maximum cyclic frequency

$\bar{f}_{S \text{ or } I}$ = Frequency in cycles per second of the desired or undesired signal

${}_1F_1(\;)$ = Hypergeometric function

θ = Phase angle

θ_I = The phase difference between the desired and the undesired signal

$G_H(\omega)$ = Power spectral density

$g(\Delta f)$ = If selectivity attenuation factor

$H(S)$ or $H(jf)$ = Filter transfer function

$h(T)$ = Impulse response of a low pass filter

$J(\;)$ = Bessel functions of the first kind

δ_{eff} = Effective duty cycle

K_{FM} = Frequency detector constant

K_{LIN} = Linear detector constant

K_{SYN} = Synchronous detector constant

K_{SQ} = Square law detector constant

K_{ϕ} = Phase detector constant

$K_Z(t)$ = The zero zone detector output convariance function

m_{Si} = The ith sub,arrier modulation index

$m_{I \text{ or } S}$ = The undesired or desired signals modulation index

$M_Z(t)$ = The zero zone detector output convariance function

N = Mean noise power

N(t) = Noise

N_o = Filtered output noise power

No_{DE} = Output noise froma de-emphasis network

n(t) = noise

η = Noise power per cycle

$O(a_o{}^6)$ = A complex function of the signal-to-noise ratio

P = Power

P_D = Probability of detecting a pulse

P_n = Power of the nth FM spectral component

p(V) = Probability density function

P(0/1) = Probability of a 0 being declared when a 1 is present

P(1/0) = Probability of a 1 being declared when a 0 is present

P(1/2) = Probability of decision in frequency channel 1 given a signal in frequency channel 2

P(2/1) = Probability of decision in frequency channel 2 given a signal in frequency channel 1

$P_{10,\ 11 \text{ or } 12}$ = Power centered at zero frequency, Δf or $2\Delta f$

R = Ratio of the input to the output signal to interference power ratio's

$RC = T$ = Low pass filter time constant

$R_I = A_I/A_S$ = Ratio of the undesired signal to the desired signal's amplitude

$R_S = A_S/A_I$ = Ratio of the desired signal to the undesired signal's amplitude

Rl( ) = Means take the real part of ( )

rms = Root mean square

rw = Weighted error rate

$\Gamma(\ )$ = Gamma function

ξ = Occupation parameter

SSB = Single sideband

$S_n(t)$ = Power spectrum

$S_K(t)$ or $I_K(t)$ = A general modulation of the desired or the undesired signal in Fourier series form

S(t) = Desired signal

$(S/I)_{O \text{ or } I}$ = The ratio of the average desired signal power to the average interference power at the output or input

$(S/I{+}N)_{o \text{ or } I}$ = The ratio of the desired signal power to the undesired signal plus noise power at the output or input

T = Time constant

T_B = Time necessary for a pulse to rise to its peak value

t = Time

V(t) = Slowly varying function compared to ω_o

V_B = Threshold detection level

V_{Bn} = Noise threshold detection level

v(t) = Narrowband signal

V_{d0} = Detector output at times at which no signal is present

V_{d1} = Detector output at times at which interference is present

$v_S(t)$ = Narrowband desired signal

$v_I'(t)$ = Narrowband undesired IF input signal

$v_I(t)$ = Narrowband undesired IF output signal

$v_d(t)$ = Detector output signal

$v_o(t)$ = Low-pass filter output signal

$X(t)$ = Even portion of the narrowband quadrature signal

$Y(t)$ = Odd portion of the narrowband quadrature signal

$<\dot{\phi}_1 \dot{\phi}_2>$ = Autocorrelation of the deterministic signal

ϕ = Phase angle

$\phi(t)$ = The ideal phase detector output

$|V(t)|$ = Envelope of the narrowband signal (a slowly varying function compared to ω_o)

$<\ >$ = Statistical average

$\overline{(\ )}$ = Time average

$\omega_{BW,LP}$ = Radian bandwidth of the ideal square low-pass filter

$\omega_{S\text{ or }I}$ = The radian frequency of the desired or undesired signal

ω_{oi} = ith subcarrier frequency

ω_o = Radian carrier frequency

σ = Standard deviation

SECTION 1

INTRODUCTION

BACKGROUND

ECAC is developing mathematical models for predicting RFI and its effects on the performance of receiving systems. This entails modeling the significant circuits in the signal path, starting from either the desired or the undesired (i.e., interfering) signal source and ending at the output of the receiving system, so that the output signal level can be predicted. Further, it is necessary to relate these predicted signal levels to the performance of the receiving system. <u>This report presents the results of a continuing effort to establish the relationship of the desired and undesired signal levels to the performance of receiving systems</u>.

Three important factors that affect the performance in a victim receiver are: the amplitude and type of modulation of the undesired signal, the amplitude and type of modulation and detection used in the desired receiving system, and the amount of frequency separation, or off-tuning, between the desired and undesired signals. Some of the effects of undesired signals for on-tune situations have already been analyzed and presented elsewhere. The investigation presented in this report is concerned with a more realistic and consequently more difficult analysis, that of off-tune situations involving different types of signal modulation and various types of detectors or receiving systems. This task obtains results for the general off-tune case as well as solutions for those on-tune cases not previously considered. The task was planned to produce analytical solutions sufficiently generalized for application to future models, as well as existing models used at ECAC.

APPROACH

The overall task of developing techniques and models for predicting the effects of undesired signals on the performance of communications receiver systems consists of four major interrelated steps: an analysis of communications systems, which leads to a measure of the degree of interference; an analysis which establishes degradation measures; the establishment of operational requirements; and the implementation of models. See Figure 1-1. The effort discussed in this report was devoted to the first of these steps, the basic system analysis problem.

The solutions obtained for the first step consist basically

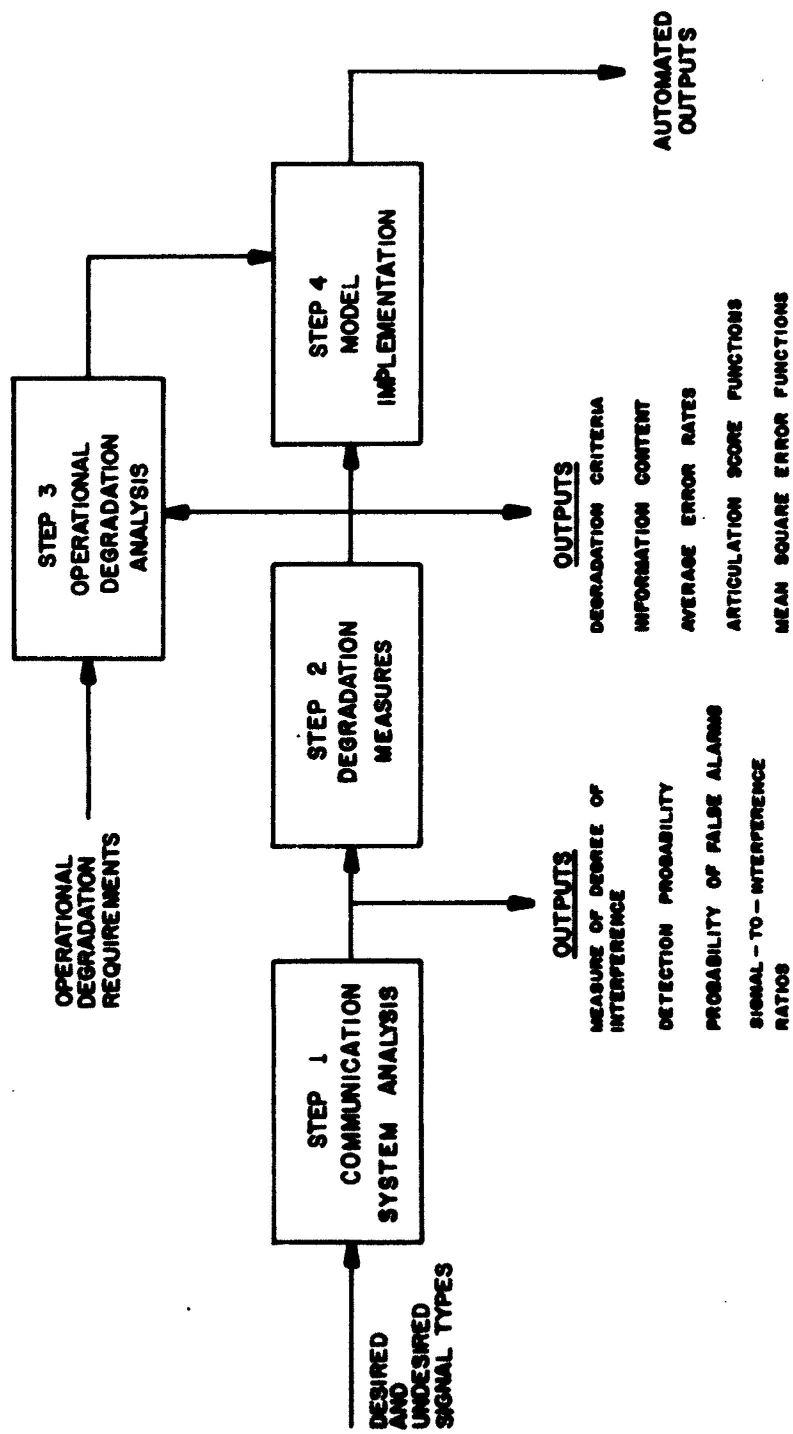


Figure 1-1. Analysis of Interference to Communications Systems

TABLE 1-1
DESIRED AND UNDESIRED SIGNAL COMBINATIONS

Desired Receiver Type \ Undesired Signal Type	AM Voice	AM Teletype	AM Analog	FM Voice	FM Teletype (FSK)	FM Analog	FDM	SSB	Pulse	Interference + Random Noise
AM Voice	X		X						X	X
AM Teletype		X							X	X
AM Facsimile	X		X						X	X
AM Digital									X	X
AM Analog	X		X						X	X
FM Voice				X		X	X		X	X
FSK					X				X	X
FM Digital									X	X
FM Analog				X		X	X		X	X
FDM				X		X	X		X	X
SSB								X	X	X
DSB-SC									X	X
Pulse									X	X
Synchronous										X
Pseudo-Noise										
Correlation										
Matched Filter										

Note: The X's denote the basic cases considered in this report.

of the detector output and the system output equations. The same detector output equations can be used for the analysis of many types of the same basic systems and consequently the results are generalized as a function of the desired and undesired system parameters such as amplitude, off-tune frequency, phase, and a generalized information signal. The basic system output equations are obtained by stating the equations representing the filtered detector output and then mathematically eliminating those variables which do not affect the final desired system answers (such as a constant phase angle for most but not all systems). It should also be pointed out that even though part of the analysis described in this report treats interference prediction in a partial deterministic manner, i.e., by assuming specific desired and undesired signal forms, this is only a development expedient and the ultimate objective is to develop statistical models designed to operate with a minimum amount of data sufficient for performance analysis.

The different combinations of modulated desired and undesired signals included in the analysis are listed in TABLE 1-1. The results of this first step have produced outputs that can be used to model the effect of undesired signals on some receiving systems. Examples of such outputs are probability of detection, probability of false alarm, and signal-to-interference ratios.

To accomplish the second step, which involves degradation analysis, it is necessary to analyze quantitatively or test the outputs of the first step to establish degradation criteria that can be applied to interference problems. Examples of degradation criteria are average error rates, articulation score functions, and mean square error criteria.

SECTION 2

RESUME OF RESULTS AND RECOMMENDATIONS

RESUME OF RESULTS

The analysis performed has produced outputs which will eventually lead to mathematical models for degradation analysis. The general conclusion reached from this initial effort is that degradation criteria can be established in a reasonable manner, using as guide lines the mathematical implications resulting from this analysis.

The analyses of communications systems were made by studying various one-to-one combinations of desired and undesired signal types. The results of this analysis are presented as formulas and are indexed in TABLE 2-1 and TABLE 2-2.

A detailed discussion of the results of each of the communication situations analyzed within this report can be found within the appropriate section under the subsection headed DISCUSSION OF RESULTS.

The specific objective of that part of the overall task discussed in this report was the analyses of a number of communication systems and consequently another objective of this overall task, degradation analysis, will not be discussed here. However, since degradation analysis remains as an objective of the overall task, a discussion of the generalization and extension of this analysis to degradation is made in SECTION 14. In particular, the considerations of Analog, Digital, Voice and Advanced Degradation Modeling should be read for general interference predictions conclusions.

RECOMMENDATIONS

Five recommendations are made:

1. This effort should be continued to complete the analysis of all desired and undesired signal combinations itemized in TABLE 1-1.
2. The analysis should be expanded to:
   a. Include the statistical effects of parameter variations (for modulation index, off tuning, and filter characteristics) to the basic performance analysis.
   b. Include multiple interference signals.
   c. Conduct a study to model special communication functions that affect the basic communication model (i.e., limiters, filters, blankers, etc.).
3. Undertake degradation analysis, as indicated in Figure 1-1.

TABLE 2-1
INDEX TO DETECTION EQUATIONS

Desired Receiver Type \ Undesired Signal	AM Voice	AM Teletype	AM Analog	FM Voice	FM Teletype (FSK)	FM Analog	FDM	SSB	Pulse	Interference + Random Noise
AM Voice	4-3 4-6	8-23	4-3 4-6	I-85 I-108	8-23	I-85 I-108	I-85 I-108	I-127 I-128	8-23	13-12
AM Teletype	I-85 I-108	11-21	I-85 I-108	8-23 I-52	11-21	I-121 I-125	8-23 I-125	I-127 I-128	11-21	11-24
AM Facsimile	4-3 4-6	8-23	4-3 4-6	I-85 I-108	8-23	I-85 I-108	I-85 I-108	I-127 I-128	8-23	13-12
AM Digital	I-85 I-108	11-21	I-85 I-108	8-23 I-52	11-21	I-121 I-125	8-23 I-52	I-127 I-128	11-21	11-24
AM Analog	4-3 4-6	8-23	4-3 4-6	I-85 I-108	8-23	I-85 I-108	I-85 I-108	I-127 I-128	8-23	13-12
FM Voice	I-186 I-206	I-186 I-206	I-186 I-206	5-2	I-188 I-206	5-2	5-2	I-179 I-206	5-2	II-22
FSK	I-186 I-206	I-186 I-206	I-186 I-206	I-186 I-206	12-30 12-31 12-34	I-188 I-206	I-188 I-206	I-179 I-206	12-30 12-31 12-34	11-24
FM Digital	I-186 I-206	I-186 I-206	I-186 I-206	I-186 I-206	12-30 12-31 12-34	I-188 I-206	I-188 I-206	I-179 I-206	12-30 12-31 12-34	11-24
FM Analog	I-186 I-206	I-186 I-206	I-186 I-206	5-2	I-188 I-206	5-2	5-2	I-179 I-206	5-2	II-22
FDM	7-5 7-29	7-5 7-29	7-5 7-29	5-2	I-188 I-206	I-188 I-206	I-188 I-206	7-5 7-29	7-5 7-29	II-22
SSB	I-234	I-234	I-234	I-220	10-9	I-220	I-220	6-1	10-9	II-24
DBS-SC	I-224b	I-231	I-224b	I-220	I-220	I-220	I-220	I-220	I-220	II-24
Pulse	----	----	----	----	----	----	----	----	----	----
Synchronous	----	----	----	----	----	----	----	----	----	----
Pseudo-Noise Correlation	----	----	----	----	----	----	----	----	----	----
Matched Filter	----	----	----	----	----	----	----	----	----	----

Note: Numbers used in this table are the equation numbers used to identify the equations in the text. Dashes indicate no analysis contained within this report.

TABLE 2-2
STEP 1*
OUTPUT EQUATIONS

Desired Receiver Type \ Undesired Signal	AM Voice	AM Teletype	AM Analog	FM Voice	FM Teletype (FSK)	FM Analog	FDM	SSB	Pulse	Interference + Random Noise
AM Voice	4-15 4-16 4-17 4-18	8-26 8-27 8-28	4-15 4-16 4-17 4-18	C	8-26 8-27 8-28	C	C	C	8-26 8-27 8-28	RC
AM Teletype	C	11-43 11-44	C	C	11-43 11-44	C	C	C	11-43 11-44	A RC
AM Facsimile	4-15 4-16 4-17 4-18	8-26 8-27 8-28	4-15 4-16 4-17 4-18	C	8-26 8-27 8-28	C	C	C	8-26 8-27 8-28	RC
AM Digital	C	11-43 11-44	C	C	11-43 11-44	C	C	C	11-43 11-44	RC
AM Analog	4-15 4-16 4-17 4-18	8-26 8-27 8-28	4-15 4-16 4-17 4-18	C	8-26 8-27 8-28	C	C	C	8-26 8-27 8-28	RC
FM Voice	RC	9-45	RC	5-7	9-45	5-7	5-7	RC	9-45	RC
FSK	C	12-45	C	C	12-45	C	C	C	12-45	RC
FM Digital	C	12-45	C	C	12-45	C	C	C	12-45	RC
FM Analog	RC	9-45	RC	5-7	9-45	5-7	5-7	RC	9-45	RC
FDM	7-38	RC	7-38	7-38	RC	7-38	7-38	7-38	RC	7-38
SSB	RC	10-17	RC	RC	10-17	RC	RC	6-6	10-17	RC
DSB-SC	RC	10-17	RC	RC	10-17	RC	RC	6-6	10-17	RC
Pulse	C	11-43 11-44	C	C	11-43 11-44	C	C	C	11-43 11-44	RC A
Synchronous	C	C	C	C	C	C	C	C	C	RC
Pseudo-Noise	RC	RC	RC	RC	RC	RC	RC	RC	RC	RC
Correlation	C	C	C	C	C	C	C	C	C	C
Matched Filter	C	C	C	C	C	C	C	C	C	C

CODE

* - See Figure 1-1 for definition of Step 1.
RC - Readily completed from the work in this report, or the work labeled "C".
A - Available in the general literature.
C - An ECAC internal technical note, which does not necessarily present the complete solution to this problem, has been completed but is not included in this report.

4. Perform tests to validate the results of the degradation analysis.

5. Accomplish model implementation of the results of the degradation analysis.

SECTION 3

COMMUNICATIONS SYSTEMS ANALYSIS

GENERAL ANALYSIS PROCEDURE

The analysis of unintentional and intentional interference on communications systems is an inherently difficult problem due to the many (n x m) different types of desired and undesired signal modulations used to transmit information. The analytical technique used in the following sections of this report, where specific interference problems are considered, is based upon the signal transfer properties of the portion of the communications receiver shown in Figure 3-1.

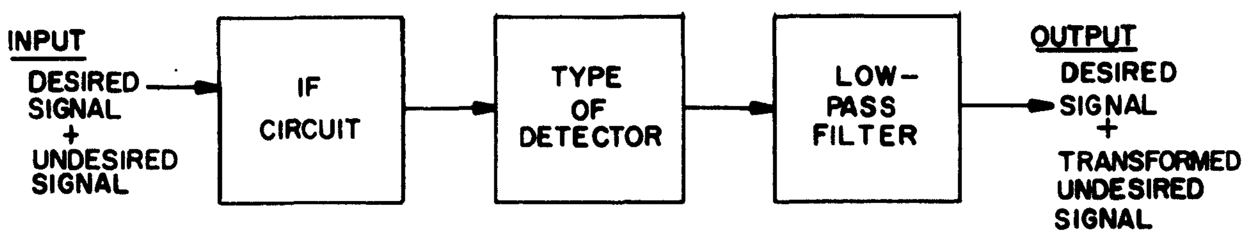


Figure 3-1. Basic Portion of Communications Receiver Analyzed for Interference Effects on a Desired Signal

In general, communications system analysis begins by considering the transformation of the desired and undesired signals by the IF circuits. For normal design conditions, the desired signal is not appreciably altered by this transformation. The effect of this transformation upon the undesired signal depends upon the type of modulation and the frequency displacement on the IF frequency transfer function or the degree of off-tuning that exists. This effect has been analytically expressed for each of the types of modulations considered and generally results in two types of transformations. The first consists of modifying the parameters of the undesired modulation while the second creates a new type of undesired modulation.

The second phase in this analysis has been to determine the effects of the detector circuit upon the combination of the desired and the modified undesired signal. The analysis at this point either obtains the bit probability of commission or omission for digital systems or continues the analysis as diagrammed in Figure 3-1 for voice and analog systems. For voice and analog systems, the output signal from the detector is separated into the desired and an interfering signal. The interfering signal at this point is a combination of the original desired and undesired signal. Although an attempt to obtain the general solution is always made, there are the cases in which the solutions can be obtained only for limiting signal conditions. The limiting signal conditions are those in which the signal level is much greater than the interference (S>>I), and those in which the interference is much greater than the signal level (S<<I). After the detector phase of analysis, the filtering effect of the low-pass filter on the output signal is analyzed.

The analysis of the filtering action of the low-pass filter has generally been accomplished by either considering an ideal boxcar-shaped low-pass frequency characteristic or a conventional RC filter. It is recognized that this approach to the solution of the problem is a simplified approximation and, if sufficient information is available for a particular problem, it is apparent that a detailed filter characteristic can be used and a modified analysis can be made analogous to those appearing in this report.

The final stage of analysis is to calculate both the power of the filtered desired and undesired input and output signal. The result is then expressed as a signal-to-interference input-to-output ratio. In the next ten sections of this report Step 1 of Figure 1-1 is discussed in detail for various desired to undesired signal combinations. After this, SECTION 14, GENERALIZATIONS AND EXTENSIONS, discusses the transition from Step 1 to degradation analysis, i.e., Step 2 of Figure 1-1.

SUMMARY

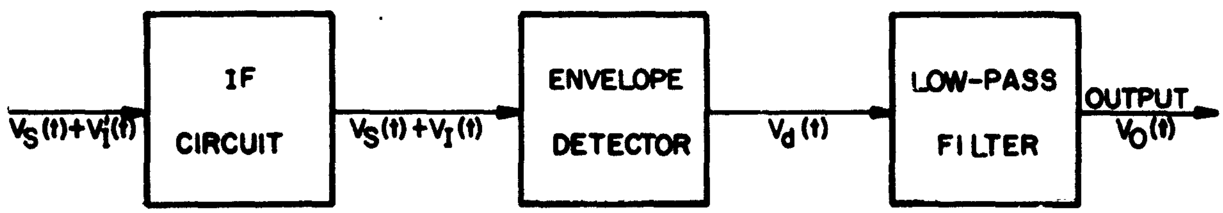
The complete solution of the many n x m communication problems is the ultimate objective of this task. The following sections contain the detailed derivations of particular signal and interference problems that have been solved. It is set up such that the general mathematical derivations are mostly contained in APPENDIX I. The directly related equations and the engineering assumptions necessary to derive the final answer for Step 1 are contained within each receiver section.

SECTION 4

A DESIRED AM SIGNAL INTERFERED WITH BY AN UNDESIRED AM SIGNAL

INTRODUCTION

The portion of the amplitude modulation (AM) receiver to be treated in the following analysis is depicted in Figure 4-1.



$v_S(t)$ = Desired signal

$v_I'(t)$ = IF input undesired signal

$v_I(t)$ = IF output undesired signal

$v_d(t)$ = Detector output signal

$v_o(t)$ = Low-pass filter output signal.

Figure 4-1. AM Receiver, Portion Analyzed

The first step in the analysis is to consider the transformation of the desired and undesired input signals by the IF circuits. For normal design conditions the desired signal is not altered by this transformation. The off-tuning of the interfering signal in general modifies the forms of the interference signal. The IF output interfering signal will, however, be considered to be of the same form as the input signal. The first order effect of the off-tuning will be taken into account by modifying the amplitude and modulation coefficients as discussed in APPENDIX III. Because of this simplification, the basic problem begins by considering the analysis of the desired and undesired

signals [$V_S(t)$ and $V_I(t)$] at the input to the detector. The mathematical aspects of this general type of problem are considered in APPENDIX I.

DETECTION AND LOW-PASS FILTER DISCUSSION

Due to the detector function, the analysis of AM is difficult to handle in a complete fashion. The detection problem can, however, be handled in the general modulation problem for the cases when the signal is much greater than the interference or when the interference is much greater than the signal. The analysis can also be performed for a general range of signal to interference, when only a single pair of information sideband components are considered. By looking at these two extremes, interference trends can be predicted. From APPENDIX I, equation (I-71a), consider the normalized AM detected output for the simple case of tone modulation.

$$
\begin{aligned}
\left.\frac{V_d(t)}{A_S}\right|_{AM} &= \frac{|V(t)|}{A_S} \\
&= 1 + \left[\frac{R_I{}^2(2 + m_I{}^2)}{8(1 - m_S{}^2)^{1/2}}\right. \\
&\quad \left. + \frac{R_I{}^4(1 + 3m_I{}^2 + 3m_I{}^4/8)(2 + m_S{}^2)}{128(1 - m_S{}^4)^{5/2}} + \cdots\right] \\
&\quad + \left[m_S - \frac{m_S R_I{}^2(2 + m_I{}^2)}{4[(1 - m_S{}^2) + (1 - m_S{}^2)^{1/2}]}\right. \\
&\quad \left. - \frac{3m_S R_I{}^4(1 + 3m_I{}^2 + 3m_I{}^4/8)}{64(1 - m_S{}^2)^{5/2}} + \cdots\right] \cos \omega_S t \\
&\quad + \left[\frac{m_I R_I{}^2}{2(1 - m_S{}^2)^{1/2}}\right. \\
&\quad \left. + \frac{m_I R_I{}^4(1 + 3m_I{}^2/4)(2 + m_S{}^2)}{32(1 - m_S)^{5/2}} + \cdots\right] \cos \omega_I t
\end{aligned}
$$

(equation continued on next page)

$$
\begin{aligned}
&+ \left[R_I - \frac{R_I^{\,3}(2 + 3m_I^{\,2})}{16(1 - m_S^{\,2})^{3/2}} + \cdots \right] \cos \Delta\omega t \\
&+ \left[\frac{m_I^{\,2} R_I^{\,2}}{8(1 - m_S^{\,2})^{1/2}} \right. \\
&\left. + \frac{m_I^{\,2} R_I^{\,4}(3 + m_I^{\,2}/2)(2 + m_S^{\,2})}{128(1 - m_S^{\,2})^{5/2}} + \cdots \right] \cos 2\omega_I t \\
&- \left[\frac{R_I^{\,2}(2 + m_I^{\,2})}{8(1 - m_S^{\,2})^{1/2}} \right. \\
&\left. - \frac{R_I^{\,4}(1 + 3m_I + 3m_I^{\,4}/8)(2 + m_S^{\,2})}{32(1 - m_S)^{5/2}} + \cdots \right] \cos 2\Delta\omega t \\
&+ \left[\frac{m_I R_I}{2} - \frac{R_I^{\,3}(12 + 3m_I^{\,2})}{64(1 - m_S^{\,2})^{3/2}} + \cdots \right] \cos (\omega_I \pm \Delta\omega)t \\
&+ \left[\frac{m_S R_I^{\,3}(2 + 3m_I^{\,2})}{16(1 - m_S^{\,2})^{3/2}} + \cdots \right] \cos (\omega_S \pm \Delta\omega)t + \cdots
\end{aligned}
$$

(4-1)
and
(I-71a)

where

A_S = the amplitude of the desired signal.

$R_I = A_I/A_S$ = the ratio of the undesired signal to the desired signal's amplitude.

$m_{I \text{ or } S}$ = the undesired or desired signal's modulation index.

$\omega_{S \text{ or } I}$ = the radian frequency of the desired or undesired signal.

$\Delta\omega$ = the radian frequency difference between the carrier of the desired and the undesired signal.

This equation represents the output signal as a number of tone modulated signals at all possible frequency combinations of the input signal ω_S, ω_I and $\Delta\omega$. The amplitude of each term is a function of an infinite series which becomes increasingly more involved for higher powers of ω. From consideration of this expression it is, however, also apparent that for most practical parameter values only a finite number and, in most cases, only the first term is needed for actual evaluation. For values of R_I less than 1/2, equation (4-1) can be approximated (with less than an estimated 10% mean square error) as

$$
\begin{aligned}
\frac{|V(t)|}{A_S} \approx\ & 1 + \frac{R_I^{\,2}(2 + m_I^{\,2})}{8(1 - m_S^{\,2})^{1/2}} + m_S \cos \omega_S t \\
& + \left[\frac{m_I R_I^{\,2}}{2(1 - m_S^{\,2})^{1/2}}\right] \cos \omega_I t + R_I \cos \Delta\omega t \\
& + \left[\frac{m_I^{\,2} R_I^{\,2}}{8(1 - m_S^{\,2})^{1/2}}\right] \cos 2\omega_I t \\
& - \left[\frac{R_I^{\,2}(2 + m_I^{\,2})}{8(1 - m_S^{\,2})^{1/2}}\right] \cos 2\Delta\omega t \\
& + \frac{m_I R_I}{2} \cos (\omega_I \pm \Delta\omega) t \\
& + \left[\frac{m_S R_I^{\,3}(2 + 3m_I^{\,2})}{16(1 - m_S^{\,2})^{3/2}}\right] \cos (\omega_S \pm \Delta\omega) t
\end{aligned}
\qquad (4\text{-}2)
$$

Next to be analyzed is the filtering action of the low-pass filter. A completely accurate answer at this point requires a detailed knowledge of the characteristics of the output filter. Since in many cases this is not available, an ideal square bandpass characteristic (that also filters the dc term) could be used. This assumption does, however, have the unrealistic form of accepting fully or rejecting the interfering signal. Therefore, the result-

ing answer, as a function of the off-tuning, exhibits a step discontinuity. In SECTION 8 this difficulty is circumvented by postulating a conventional RC filter given by equation (8-7). Unfortunately, this adds to the details of the calculations and therefore, for the present section, the ideal low-pass filter will be postulated. If sufficient information is available on a particular type of communication equipment, it is apparent that the detailed filter characteristic can readily be used and may therefore be incorporated in future work for specific filter characteristics. For the square or boxcar filter characteristics, the processing of equation (4-1) becomes one of deciding whether or not the frequency of interest is above or below the high frequency cut-off limit.

The following analysis applies to low and high level modulation index for analog as well as voice modulation, although the discussion will normally be concerned with voice modulation.

For AM voice modulation it is not desirable to allow the instantaneous normalized amplitude to exceed one and therefore produce excessive distortion. Noise is generally considered to have a reasonably small probability of exceeding from 3 to 4 times its root mean square value. Since the statistical variations of voice are associated with random noise, it is in many cases common to design around modulation indexes of between .25 and .33. A typical value of .3 is therefore often used for analysis. For voice modulation it is also possible to simplify equation (4-2) further. Considering small values of modulation index, the relative magnitude of the interfering terms, and neglecting the terms which are usually filtered out we obtain from equation (4-2) the further simplification.

$$\frac{|V(t)|}{A_S} \approx m_S \cos \omega_S t + (m_I R_I^2/2)\cos \omega_I t + R_I \cos \Delta\omega t + (m_I R_I/2)\cos (\omega_I \pm \Delta\omega)t \qquad (4\text{-}3)$$

when, $\Delta\omega \le \omega_{BW,LP}$

$\omega_{BW,LP}$ = bandwidth of the ideal square low-pass filter.

The output signal therefore ideally consists of the desired intelligence term, the undesired intelligence term and a beat interfering term. When $\Delta\omega > \omega_{BW,LP}$* the output signal is obtained from equation (4-3) as

*and also when $\omega_I - \Delta\omega > \omega_{BW,LP}$.

$$\frac{|V(t)|}{A_S} \approx m_S \cos \omega_S t + \frac{m_I R_I^{\,2}}{2} \cos \omega_I t \qquad (4\text{-}4)$$

It is interesting at this point to stop the systematic analysis trend and consider a second method of deriving equation (4-1). The introductory discussion mentioned the fact that for the special case of a large signal-to-interference ratio or a large interference-to-signal ratio a general answer could be obtained. From APPENDIX I, equation (I-85), the detector output for a general amplitude modulated desired and undesired signal is obtained from a series solution as

$$\begin{aligned}\frac{|V(t)|}{A_S} \approx\ & 1 + S_K(t) \\ & + .75R_I[1 + S_K(t) + I_K(t) + S_K(t)\cdot I_K(t)]\cos(\Delta\omega t+\theta_I) \\ & + .75R_I^{\,2} I_K(t) \qquad (4\text{-}5)\end{aligned}$$

where

$S_K(t)$ or $I_K(t)$ = the general modulation of the desired or the undesired signal in Fourier series form.

It is apparent upon comparing equations (4-2) and (4-5) that they are similar for tone modulation, except for the .75 beat tone amplitude. This smaller value is due to the partial series used in the derivation of equation (4-5). Except for this fact, and the restriction of a large signal-to-interference ratio, a general AM modulated signal could have been hypothesized. For voice signals it is probably sufficient to use tone modulation and hence the previous derivations are adequate. It may, however, be necessary in the future for degradation analysis (step 2 of Figure 1-1) to set up the relationship in terms of general modulated signals, and then apply equation (4-5) and the series method.

Since equation (4-1) was obtained for the case when A_S is greater than A_I, a second set of relationships is also needed for the case of A_I being greater than A_S. For this case the same basic equation applies, but the output should be re-expressed by replacing S with I as

$$\frac{|V(t)|}{A_I} \approx m_I \cos \omega_I t + (m_S R_S{}^2/2) \cos \omega_S t + R_S \cos \Delta\omega t + (m_S R_S/2) \cos (\omega_S \pm \Delta\omega)t \tag{4-6}$$

when $\Delta\omega \leq \omega_{BW,LP}$

and

$$\frac{|V(t)|}{A_I} \approx m_I \cos \omega_I t + (m_S R_S{}^2/2) \cos \omega_S t \tag{4-7}$$

when

$$\Delta\omega > \omega_{BW,LP}$$

This completes the detector and the low-pass filter part of the AM problem for a voice modulated case under assumed modulation conditions.

It is now desired to calculate the power transfer equation. The desired and undesired input signals for amplitude modulation are given by

$$S(t) = A_S(1 + m_S \cos \omega_S t) \cos \omega_o t \tag{4-8}$$

and

$$I(t) = A_I(1 + m_I \cos \omega_I t) \cos [(\omega_o + \Delta\omega)t + \theta_I] \tag{4-9}$$

The average S/I power ratio at the input is therefore obtained as

$$(S/I)_I = \frac{A_S{}^2/2 + A_S{}^2 m_S{}^2/4}{A_I{}^2/2 + A_I{}^2 m_I{}^2/4}$$

$$= R_S{}^2 \left(\frac{2 + m_S{}^2}{2 + m_I{}^2} \right) \tag{4-10}$$

where $R_S = A_S/A_I$

The S/I at the output is determined by the four cases previously discussed. By using equations (4-3), (4-4) and (4-6),(4-7), the following four cases are readily derived.

CASE 1.

$(S >> I;\ \Delta\omega \leq \omega_{BW,LP})$

$$(S/I)_o = \frac{4m_S{}^2}{R_I{}^2(4 + m_I{}^2R_I{}^2 + m_I{}^2)} \qquad (4\text{-}11)$$

CASE 2.

$(S >> I;\ \Delta\omega > \omega_{BW,LP})$

$$(S/I)_o = \frac{4m_S{}^2}{m_I{}^2R_I{}^4} \qquad (4\text{-}12)$$

CASE 3.

$(I >> S;\ \Delta\omega \leq \omega_{BW,LP})$

$$(S/I)_o = \frac{m_S{}^2R_S{}^2}{4(1 + R_I{}^2m_I{}^2)} \qquad (4\text{-}13)$$

CASE 4.

$(I >> S;\ \Delta\omega > \omega_{BW,LP})$

$$(S/I)_o = \frac{m_S{}^2R_S{}^4}{4m_I{}^2} \qquad (4\text{-}14)$$

The ratio of the S/I at the input to the S/I at the output is therefore obtained by dividing equations (4-11) to (4-14) into equation (4-10)*. The resultant answers are obtained as:

CASE 1.

$$R = \frac{(S/I)_I}{(S/I)_o} = \frac{(2 + m_S^2)(4 + m_I^2 R_I^2 + m_I^2)}{(2 + m_I^2)(4m_S^2)} \tag{4-15}$$

CASE 2.

$$R = \frac{(2 + m_S^2)(m_I^2 R_I^2)}{(2 + m_I^2)(4m_S^2)} \tag{4-16}$$

---

*It should be noted at this stage of the analysis that the representation of the ratio R leads to certain mathematical difficulties if the results are used without regard to the manner in which they were derived. In particular, if it is assumed that the input (and consequently the output) interference values are zero, the ratio R becomes indeterminate in the form of $\frac{0}{0}$. Since all the results for R are derived on the basis that an interfering signal is present, this substitution is not valid. If R is not used, and the form

$$(S/I)_I = (S/I)_o \, f(R_I, m_I, m_S)$$

is used the resulting answer for the case of zero interference becomes that of $\infty = \infty$. Although this is mathematically more satisfiable than the previous formulation, the resulting answer is still no more useful. The ratio could also be written in the form $R' = (I/S)_I / (I/S)_o$. Although this would circumvent the difficulty obtained when I goes to 0, the form is not the most useful, since for most systems intelligibility can only practically be conveyed when $(I/S)_o$ is a fractional value and consequently R' is a small fractional value. Since positive whole numbers are generally preferred, this form will also be rejected. A final comment is that the ratio R is basically presented for its symbolic simplification and should, consequently, be used in this manner.

CASE 3.

$$R = \frac{4(2 + m_S^2)(1 + R_I^2 m_I^2)}{(2 + m_I^2)(m_S^2)} \qquad (4\text{-}17)$$

CASE 4.

$$R = \frac{4(2 + m_S^2)(R_I^2 m_I^2)}{m_S^2(2 + m_I^2)} \qquad (4\text{-}18)$$

A case of special interest for the same type of desired and undesired signals is when $m_I = m_S = m$. For this case the four outputs are given by:

CASE 1.

$$R = \frac{4 + m^2 R_I^2 + m^2}{4m^2} \qquad (4\text{-}19)$$

CASE 2.

$$R = (R_I^2/4) \qquad (4\text{-}20)$$

CASE 3.

$$R = \frac{4(1 + m^2 R_I^2)}{m^2} \qquad (4\text{-}21)$$

CASE 4.

$$R = 4R_I^2 \qquad (4\text{-}22)$$

For general prediction purposes it is further desired to program typical parameter values of the output equation. For this particular case only the modulation index is affected.

A typical value of the modulation index (m) was previously discussed and is given by m = .3. For this value, the four output equations reduce to:

CASE 1.

$$R = (45 + R_I^2)/4 \tag{4-23}$$

CASE 2.

$$R = R_I^2/4 \tag{4-24}$$

CASE 3.

$$R = 4(11 + R_I^2) \tag{4-25}$$

CASE 4.

$$R = 4R_I^2 \tag{4-26}$$

DISCUSSION OF RESULTS

These output ratios are shown in Figure 4-2 for various values of R_I when $\Delta\omega \leq \omega_{BW,LP}$ and in Figure 4-3 when $\Delta\omega > \omega_{BW,LP}$. They are shown both for the case when m = .3 and also for the case when m = 1*. It is apparent from a consideration of equations (4-20) and (4-22) or Figure 4-2 that in all cases the filtered region

*Using m = 1, is not strictly mathematically correct. See equation (4-2) and (I-69).

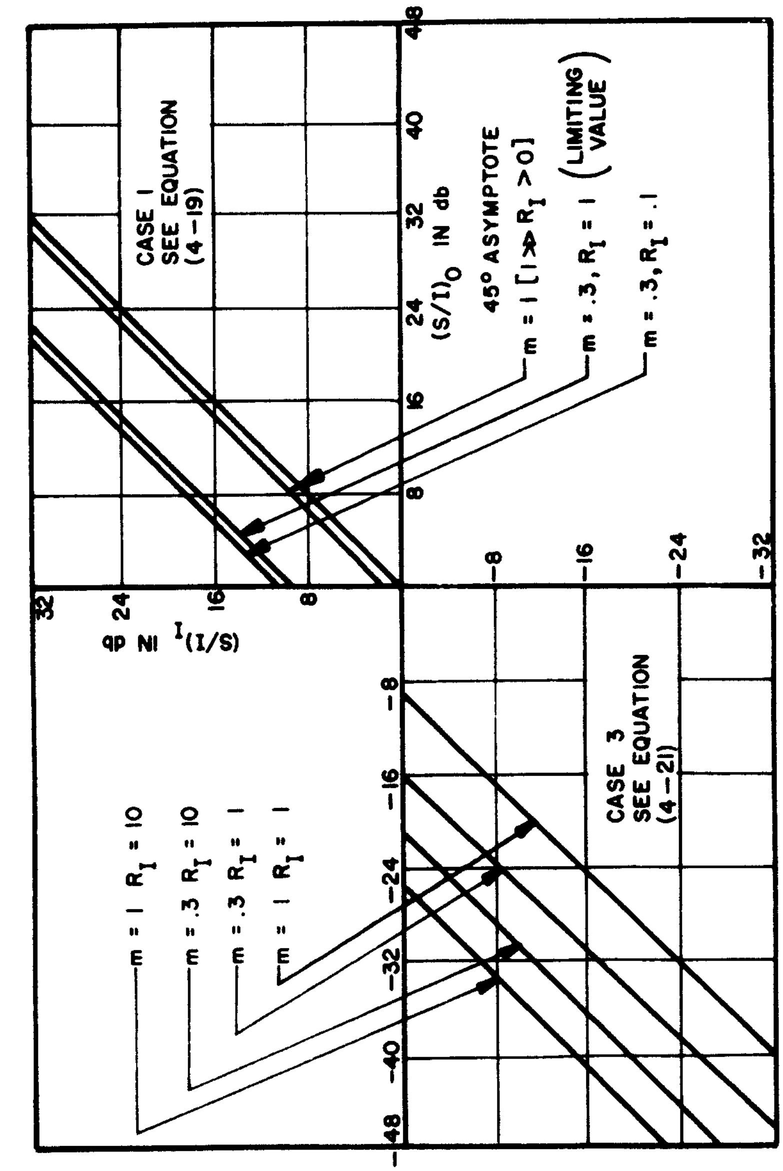


Figure 4-2. AM Desired vs AM Undesired Signal ($\Delta\omega \leq \omega_{BW-LP}$)

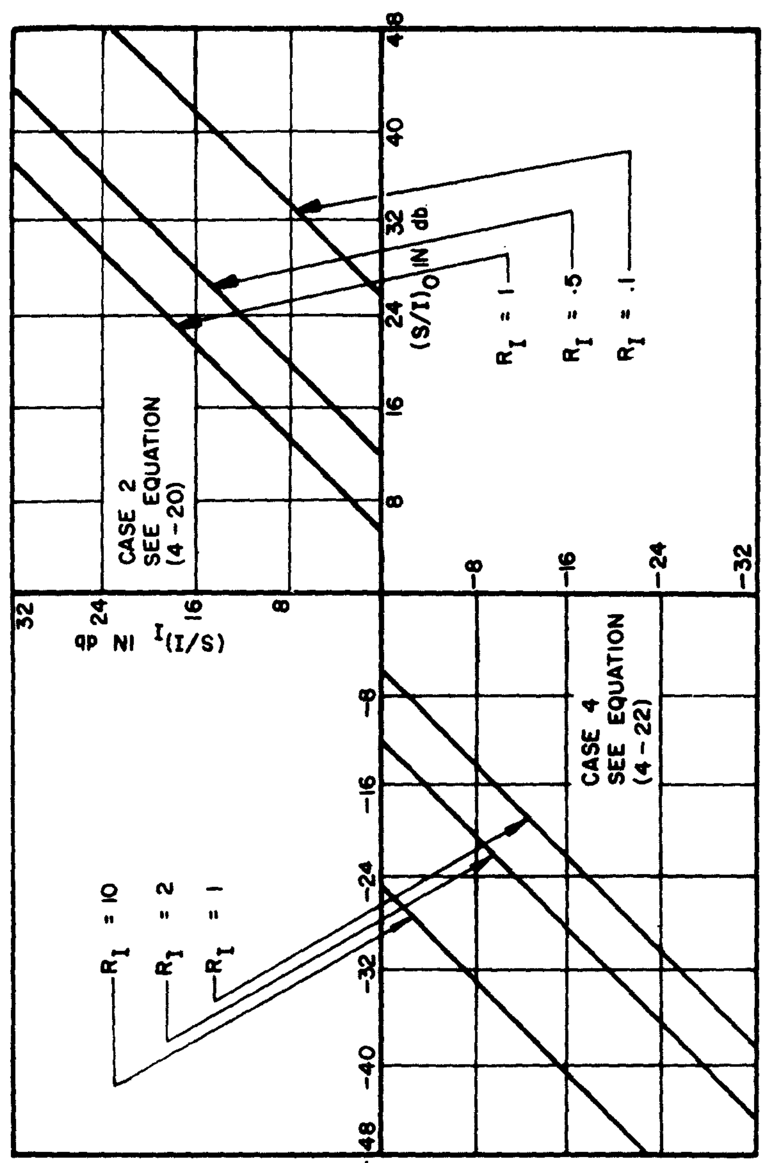


Figure 4-3. AM Desired Signal vs AM Undesired Signal ($\Delta\omega > \omega_{BW,LP}$)

($\Delta\omega > \omega_{BW,LP}$) is directly dependent on the carrier ratio R_I. The unfiltered signal-to-interference region for ($\Delta\omega \leq \omega_{BW,LP}$) is, however, dependent on the carrier ratio as well as the modulation index. The overall implication that can be drawn from this analysis is that, for AM, the carrier ratio is the key determinant factor in system performance.

The basic problem remaining for degradation analysis is to obtain the intelligibility functions corresponding to the four cases. The type of functions that should be obtained are determined by the type of terms in the output equations and consequently the real signals they represent. That is, if the output signal contains a beat signal term, a beat tone intelligibility test should be made or obtained. A table representing the connection between the type of intelligibility tests and the four cases is given in TABLE 4-1. Some of these intelligibility relationships exist while others need to be obtained. The problem of handling combinations of intelligibility types needs investigation to determine a simple method of obtaining the combination intelligibility functions without testing all possible combinations of the parameter values.

Although the latter part of this section specifically discussed voice modulation, the resulting equations obtained for the low and high level modulation index cases also apply to analog modulation.

TABLE 4-1
AM ACCEPTABILITY CASES

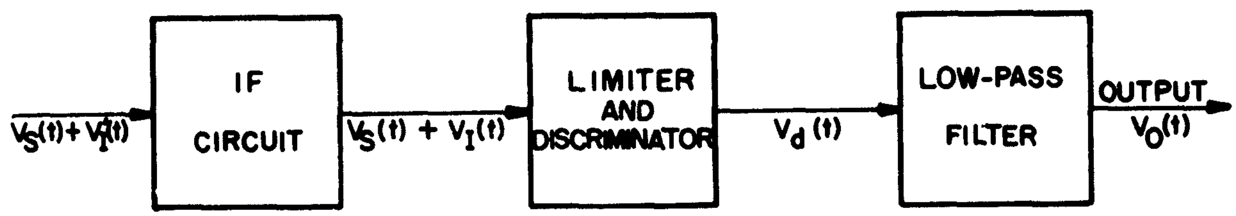
Type of Interference	Parameter	Output Equation
Voice Babble Plus Tone at $\Delta\omega$	$S > I$ $\Delta\omega \leq \omega_{BW,LP}$	CASE 1. Equation (4-3)
Voice Babble	$S > I$ $\Delta\omega > \omega_{BW,LP}$	CASE 2. Equation (4-4)
Voice Babble Plus Tone at $\Delta\omega$	$I > S$ $\Delta\omega \leq \omega_{BW,LP}$	CASE 3. Equation (4-6)
Voice Babble	$I > S$ $\Delta\omega > \omega_{BW,LP}$	CASE 4. Equation (4-7)

SECTION 5

A DESIRED FM SIGNAL INTERFERED WITH BY AN UNDESIRED FM SIGNAL

INTRODUCTION

The portion of the frequency modulated (FM) receiver to be treated in the following analysis is depicted in Figure 5-1.



where $v_S(t)$ = Desired Signal

$v_I'(t)$ = IF input undesired signal

$v_I(t)$ = IF output undesired signal

$v_d(t)$ = Detector output signal

$v_o(t)$ = Low-pass filter output signal.

Figure 5-1. FM Receiver, Portion Analyzed

The first step in the analysis is to consider the transformation of the desired and undesired input signals by the IF circuits. For normal design conditions the desired signal is not altered by this transformation. The off-tuning of the interfering signal in general modifies the form of the interference signal. For FM systems the IF is, however, designed to have little or no fall off within the region of linear discrimination or ideal detection. The interfering signal is therefore only changed by the linear phase characteristics of the IF. APPENDIX III discusses that this results in only a linear phase shift of the interfer-

ence intelligence signal. The interference signal from the IF is consequently of the same form as the input signal. The problem at this point is therefore to consider the ideal combination of a desired and undesired off-tuned FM signal.

The desired signal will also be assumed to be operating in the normal design ranges with the signals being fully limited. This, then, eliminates all amplitude variations of the signal and only the ideal discriminator output need be considered.

DETECTION AND LOW-PASS FILTER DISCUSSION

The analysis of the FM interference problem is more difficult to evaluate than the AM problem, since phase and/or frequency detectors produce a series of harmonics of the input frequencies whose amplitude cannot be neglected in the manner that was done for AM when equation (4-2) was obtained from equation (4-1). However, the starting point or the type of analysis is of the same principle as the AM problem. It is therefore desired to obtain the detector output. From APPENDIX I, equation (I-188)*we directly obtain for the general case

$$V_d(t)\Big|_{FM} = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = \frac{\frac{d}{dt}[S_K(t)]}{2\pi} + \frac{R_I}{2\pi}\left\{\Delta\omega + \frac{d}{dt}[I_K(t)] - \frac{d}{dt}[S_K(t)]\right\}\left\{\frac{\cos[\Delta\omega t + \theta_I + I_K(t) - S_K(t)] + R_I}{1 + R_I^2 + 2R_I\cos[\Delta\omega t + \theta_I + I_K(t) - S_K(t)]}\right\} \quad (5\text{-}1)$$

where $\phi(t)$ = the ideal phase detector output

This form of the detected output signal is not convenient for filter calculation and it is therefore desired to reformulate this in a series form. The derivation of this equivalent form is considerably involved and is given in APPENDIX I for tone modulation. From APPENDIX I equation (I-206) the FM output can be rewritten as:

$$V_d(t)\Big|_{FM} = \frac{1}{2\pi}\frac{d\phi(t)}{dt} = f_S B_S \cos\omega_S t - I_0(t) \quad (5\text{-}2)$$

*where the definition of $S_K(t)$ has been slightly changed; also see (I-180).

where

$$I_o(t) = \sum_{n=1}^{\infty} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (-R_I)^n \left[\frac{rf_I}{n} - \frac{sf_S}{n} + \Delta f\right] J_S(nB_S) J_r(nB_I) \cos (n\Delta\omega t + n\theta_I + r\omega_I t - s\omega_S t) \qquad (5\text{-}3)$$

$B_{S \text{ or } I}$ = frequency modulation index of the desired or undesired signal

$f_{S \text{ or } I}$ = frequency in cycles per second of the desired or undesired signal

$J(\ )$ = Bessel functions of the first kind.

It is apparent that because of the complicated form of the output interfering signal it would not be practical to attempt to find the series equivalent form for a general information modulated signal $[S_K(t)]$. Although the output signal is somewhat complex, the calculation of the filtered output for a boxcar filter is straightforward. In particular, this involves limiting the upper bounds on the summation indices.

For the particular case of a boxcar filter and a single modulating frequency [f_I or f_S of equation (5-3) is set equal to zero] the asymptotic value of $I_o(t)$ reduces to:

$$I_o'(t) = \sum_{0}^{r'} (-R_I) [rf_I + \Delta f] J_r(B_I) \cos (\Delta\omega t + r\omega_I t) + \sum_{0}^{r'} (-R_I) [\Delta f - rf_I] J_{-r}(B_I) \cos (\Delta\omega t - r\omega_I t) \qquad (5\text{-}4)$$

where

$$r' = \frac{\omega_{BW,LP} - \Delta\omega}{\omega_I}$$

Since the output signal has been reduced to a sum of harmonic components, the calculation of the average interfering output power reduces to squaring the magnitude of each component and dividing by one half. This part of the problem has, therefore,

been formulated for hand calculation.

For the next step in the analysis, the input FM signal-to-interference ratio is readily found to be:

$$(S/I)_I = \frac{(A_S{}^2/2)}{(A_I{}^2/2)} = R_S{}^2 \tag{5-5}$$

The desired signal's output power for tone modulation is obtained from equation (5-1) as

$$S_o = (\omega_S{}^2\ B_S{}^2/2) \tag{5-6}$$

The desired ratio is therefore obtained as

$$R = \frac{R_S{}^2 \cdot \overline{I_o{}^2(t)}}{(\omega_S{}^2 B_S{}^2/2)} = 2(R_S/\omega_S B_S)^2 \cdot \overline{I_o{}^2(t)} \tag{5-7}$$

where

$\overline{I_o{}^2(t)}$ symbolizes the average power obtained from the summation process indicated by equation (5-3) or (5-4).

DISCUSSION OF RESULTS

Equation (5-7) is shown for various parameter values in Figure 5-2. This set of curves is adequate for analog systems where tone modulation is a sufficient intelligibility representation.

The basic interference suppression property of FM is evident by the considerable increase in the signal-to-interference power ratio between the input and the output of the basic system.

Figure 5-1 is only a representation of the positive signal-to-interference space, and therefore a representation of the negative region is also required. FM systems, however, have such a sharp cutoff or suppression of the desired signal for ratios of $A_I > A_S$ (the capture effect), that the desired signal is almost completely suppressed. This can be seen by a direct examination of equation (5-3) where the desired and undesired signal's nomen-

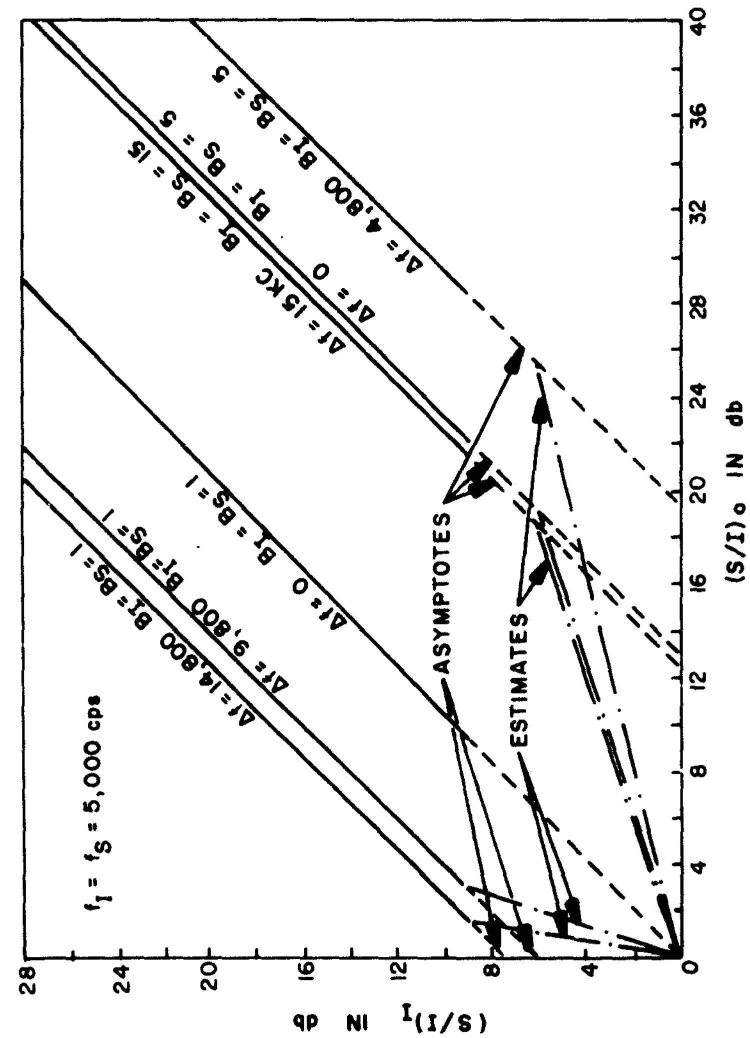


Figure 5-2. FM Desired Signal vs FM Undesired Signal

clature have been interchanged. From this equation it is evident that the desired term [when $I_o(t)$ has been changed to $S_o(t)$] has been completely scrambled by the undesired signal. A simpler explanation can be obtained by examining the large interference-to-signal relationship that can basically be obtained from equation (5-1). (See for an example of this type of approximation I-169.) The frequency detector output is readily obtained as:

$$\frac{d\phi(t)}{dt} = \frac{d}{dt}[I_K(t)] + R_S\{\Delta\omega + \frac{d}{dt}[S_K(t)] - \frac{d}{dt}[I_K(t)]\} \cos [\Delta\omega t + \theta_I + S_K(t) - I_K(t)] \quad (5\text{-}8)$$

when

$$A_I >> A_S$$

The desired signal is therefore obtained as

$$|S_o(t)| \approx R_S S_K(t) \quad (5\text{-}9)$$

which is only obtained when

$$\Delta\omega + \theta_I + S_K(t) - I_K(t) = K\pi \qquad K = 0,1, \ldots K \quad (5\text{-}10)$$

The analysis of the conditions necessary to meet this requirement is a statistical problem, depending upon the structure of $S_K(t)$ and $I_K(t)$. It is apparent, however, that the probability of equation (5-10) becoming zero and remaining zero for a sufficient length of time to convey information would be extremely small. It is for this reason that the negative signal-to-interference space will be considered zero. A number of basic problems remain in the FM voice analysis region. Some of these are: a discussion of a general modulated signal, the coupling of the intelligibility functions to equation (5-2), and a normal FM roll-off filter.

A general FM modulation output can be obtained for a large signal-to-interference approximation and produces the simpler form of equation (5-1) given by

$$\frac{d\phi(t)}{dt} = \frac{d}{dt}[S_K(t)] + R_I\{\Delta\omega + \frac{d}{dt}[I_K(t)] - \frac{d}{dt}[S_K(t)]\} \cos [\Delta\omega t + \theta_I + I_K(t) - S_K(t)] \quad (5\text{-}11)$$

when

$R_I << 1.$

A series expansion of the cos ( ) portion is again required. The important point in this discussion is that the large signal-to-interference approximation allows a solution for reasonably general modulated signals while a general amplitude solution would be prohibitively detailed.

Either from consideration of equations (5-2) or (5-3) the interference signal for B_S or $B_I >> 1$ is seen to be spread throughout the desired signals band. This is an inherent property of frequency and phase modulated systems. The equations as they presently stand are equally representative of voice or analog modulation. The next problem is to connect this output signal with the appropriate intelligibility functions. The following discussion will be devoted basically to an intelligibility discussion for voice modulated signals.

It is apparent that for the simpler but representative case of tone modulation the interference consists of tone modulated signals at the discrete frequencies given by equation (5-3). If a general attempt was made to obtain intelligibility curves representing all possible variations of this equation a large number of curves would be required. There are, however, two areas that can be modeled. These are the areas in which only a few harmonics fall within the intelligence pass band and the case in which a large number fall within the band. For the first case a test could be set up and articulation scores or intelligibility measured. For the second case evidence and logic indicate (see reference 21)that as a number of components become large the degradation approaches that obtained from noise. Although this is a reasonable procedure for wide band frequency and phase detectors, further examination is required to determine the exact limits for both approaches.

The final problem is to consider a normal FM roll-off filter. Changing the requirements or shape of the output filter obviously changes the amount of output interference power. An examination of this topic is also required because the FM low-pass filter output stage has a definite de-emphasis characteristic to com-

pensate for the transmitted pre-emphasized signal. In particular, a time constant of 75 μseconds is commonly used for broadcast reception. A simple low-pass de-emphasis network is shown in Figure 5-3. The transfer function of this network is readily obtained as

$$\left|\frac{E_o}{E_I}\right|^2 = \frac{1}{1 + \omega^2 \tau^2} \tag{5-12}$$

Since the calculation for discrete components reduces to a laborious calculation involving equation (5-4), a simpler approach is desired. Consider, instead, the limiting condition, or one where the interference can be considered as noise. It was determined in APPENDIX II, equation (II-22), that the average power per frequency from an FM system for a large carrier to noise condition is given by

$$\overline{\left[\frac{d\phi(t)}{dt}\right]^2} = \frac{\omega_M{}^2 N}{A_S{}^2} \tag{5-13}$$

where

N = mean noise power.

The output filtered noise power for a bandwidth filter of width B is given by

$$N_{o_{DE}} = 2\left(\frac{4\pi^2 N}{A_S{}^2}\right) \int_0^B \frac{f^2}{1 + 4\pi^2 \tau^2 f^2} \, df \tag{5-14a}$$

$$= \frac{8\pi^2 N}{A_S{}^2} \left[(B/4\tau^2\pi^2) - (1/8\tau^3\pi^3) \tan^{-1}(2\pi B\tau)\right] \tag{5-14b}$$

For the previously described case of an ideal square cutoff filter of width B the output power is obtained as

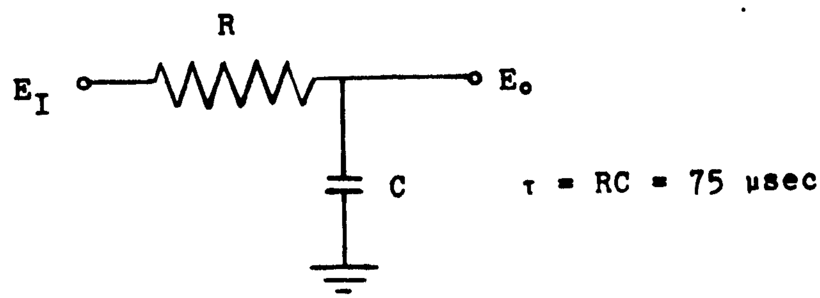


Figure 5-3. Low-Pass De-Emphasis Network

$$N_o = 2\left(\frac{4\pi^2 N}{A_S^{\,2}}\right)\int_0^B f^2\,df = \frac{8\pi^2 N}{A_S^{\,2}}\left[\frac{B^3}{3}\right] \tag{5-15}$$

The $\tan^{-1}(x)$ in equation (5-14b) can be expanded in the series

$$\tan^{-1}(x) = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^2} - \ldots \tag{5-16}$$

Using the first two terms of this series (valid for large x) equation (5-14) can be reduced to

$$N_{o_{DE}} = \frac{8\pi^2 N}{A_S^{\,2}}\left[\frac{B}{4\tau^2\pi^2} - \frac{1}{8\tau^3\pi^3}\left(\frac{\pi}{2} - \frac{1}{2\pi B\tau}\right)\right] \tag{5-17}$$

As B becomes large this can be approximated by

$$N_{o_{DE}} \underset{B\to\infty}{\approx} \frac{8\pi^2 N}{A_S^{\,2}}\left[\frac{B}{4\tau^2\pi^2}\right] \tag{5-18}$$

The ratio of the de-emphasized output noise power to the box-car-filtered output noise is obtained from equations (5-15) and (5-18) as

$$\frac{N_{o_{DE}}}{N_o} = \frac{3}{4\pi^2(\tau B)^2} \approx \frac{1}{13.2(\tau B)} \tag{5-19}$$

When $\tau B = 1$ this becomes

$$\frac{N_{o_{DE}}}{N_o} = \frac{1}{13.2} \ (-11.2 \text{ db}) \tag{5-20}$$

This, in turn, indicates that in the limit of a large number of CW interfering components there will be a resulting difference of 11.2 db between the two answers. An approach to adopt is therefore to use the 11.2 db correction factor for this case and to use the derivation in equations (5-4) and (5-12) for a small number of interfering components.

The number of components that must be practically considered depends upon $\Delta\omega$, ω_S and ω_I. The frequency difference, $\Delta\omega$, depends upon an individual problem, but the values of ω_S and ω_I can be chosen by considering the type of output that is desired. Basically, intelligibility versus an average power signal-to-interference output is desired and for this output a possible choice is the worst interfering frequencies for ω_S and ω_I. A value based on the maximum value of ω_S and ω_I for the frequency range of interest, could therefore be chosen. A second approach would be to use the centroid frequency value of the desired audio power spectrum. A final comment about the choice of the frequency (ω) is that in testing systems a value of 1000 cps is commonly chosen. While this is a reasonable value to use for system testing it does not necessarily have the direct bearing on intelligibility that the previously discussed values have and, probably, should not be used for intelligibility and consequently degradation analysis.

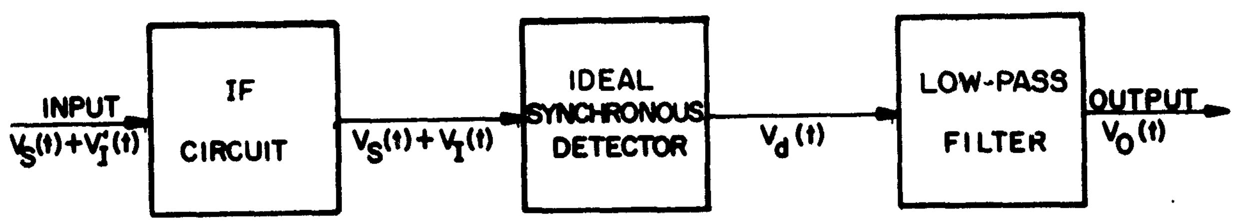
It should again be mentioned that the basic derivations of this section are equations (5-2) or (5-3), which apply equally well to voice or analog FM modulation.

SECTION 6

A DESIRED SSB SIGNAL INTERFERED WITH BY AN UNDESIRED SSB SIGNAL

INTRODUCTION

The portion of the single sideband (SSB) receiver to be treated in the following analysis is depicted in Figure 6-1.



$v_S(t)$ = Desired signal

$v_I'(t)$ = IF input undesired signal

$v_I(t)$ = IF output undesired signal

$v_d(t)$ = Detector output signal

$v_o(t)$ = Low-pass filter output signal.

Figure 6-1. SSB Receiver, Portion Analyzed

The first step in the analysis is to consider the transformation of the desired and undesired input signals by the IF circuits. For normal design conditions the desired signal is not altered by this transformation. The off-tuning of the interfering signal in general modifies the form of the interference signal. The IF output interference signal will, however, be considered to be of the same form as the input signal. The first order effect of the off-tuning will subsequently be taken into account by modifying the amplitude and modulation coefficients as

discussed in APPENDIX III. Because of this simplification the basic problem begins by considering the analysis of the desired and undesired signals [$V_S(t)$ and $V_I(t)$] at the input to the detector.

DETECTOR AND LOW-PASS FILTER DISCUSSION

The ideal synchronous detector produces a linear translation of the combined input signal. The detected output desired and undesired signal is readily found in APPENDIX I, equation (I-220), to be

$$V_d(t)\Big|_{SYNC} = \frac{X(t)}{2} = \frac{A_S}{2}\sum_{K=1}^{N} m_{SK}\cos(\omega_{SK}t)$$
$$+ \frac{A_I}{2}\sum_{K=1}^{N} m_{IK}\cos[(\omega_{IK} + \Delta\omega)t + \theta_I] \qquad (6\text{-}1)$$

The detected output interference therefore consists of the input signal located at the beat frequency Δf and changed in phase (relative to the desired signal) by the phase difference (θ_I) between the reference oscillation and the interference signal.

The problem now becomes one of applying filter analysis to the interfering signal. The ideal bandpass or that of an RC filter could be used. The ideal bandpass example was discussed in SECTION 4 and could be similarly applied here. The RC filter analysis will be used in SECTION 10 and this analysis could also directly be applied here except for the change in calculating the interference modulation index. However, due to the sharp selectivity curves for modern SSB receivers, the most instructive calculation for a synchronous system is simply the power ratio, R. This can be readily obtained.

The output signal for tone modulation is found from equation (6-1) to be

$$V_o(t) = \frac{A_S m_S}{2}\cos(\omega_S t + \theta_S)$$
$$+ \frac{A_I m_I}{2}\cos[(\omega_I + \Delta\omega)t + \theta_I] \qquad (6\text{-}2)$$

The output signal to interference power ratio is

$$(S/I)_o = R_S{}^2\left(\frac{m_S{}^2}{m_I{}^2}\right) \tag{6-3}$$

The input interfering signal is given by

$$I(t) = A_I \sum_{K=1}^{N} m_{IK} \cos\left[(\omega_{IK} + \Delta\omega + \omega_o)t + \theta_I\right] \tag{6-4a}$$

$$\begin{aligned} I(t) = {} & A_I \sum_{K=1}^{N} m_{IK} \cos\left[(\omega_{IK} + \Delta\omega)t + \theta_I\right] \cos \omega_o t \\ & - A_I \sum_{K=1}^{N} m_{IK} \sin\left[(\omega_{IK} + \Delta\omega)t + \theta_I\right] \sin \omega_o t \end{aligned} \tag{6-4b}$$

The input signal-to-interference ratio for tone modulation is

$$(S/I)_I = \frac{A_S{}^2 m_S{}^2/2}{A_I{}^2 m_I{}^2/2} = R_S{}^2\left(\frac{m_S{}^2}{m_I{}^2}\right) \tag{6-5}$$

The input-to-output ratio is consequently

$$R = \frac{(S/I)_I}{(S/I)_o} = 1, \quad (0\ \text{db}) \tag{6-6}$$

when

$$\Delta\omega \leq \omega_{BW} .$$

DISCUSSION OF RESULTS

The basic transfer function obtained from equation (6-6) is one of 45° slope going through 0 db and will not be reproduced here due to its simplicity. This is derived without considering the filtering effects of the low-pass filter, which would reduce

the interfering signal proportional to the off-tuning. The low-pass filter can be taken into account by a step function as in the previous sections or by a conventional RC filter as in subsequent sections. In particular it can be seen from SECTION 10 that for an RC filter the relationship of equation (10-14), in which δ represents the rms equivalent undesired SSB signal, and equation (10-17) allows Figure 10-1 to be applied to the off-tuning problem of this section.

Due to the linear translation property of SSB systems, a discussion of intelligibility is far simpler than in previous sections. The basic point is that, due to the linear translation, the form of the intelligibility is not changed by the detector, providing the signal is not off-channel. Any testing, therefore, requires only the use of the same type of off-tuned intelligibility signal as was applied at the system input.

SECTION 7

THE DESIRED TO UNDESIRED SIGNAL PERFORMANCE OF MULTIPLEX SYSTEMS

INTRODUCTION

The purpose of this section is to derive the expression for the average value of the detected output signal appropriate for the calculation of the desired-to-undesired signal ratio for specific types of multiplex systems. Generally, these derivations are divided into the categories of analog or voice and digital modulation. The conversion from the average value calculation associated with analog or voice modulation to the peak value calculation associated with digital modulation is a relatively straightforward procedure. Therefore, only the basic analog or voice modulation case will be discussed. The analog category will, in turn, be further divided into the main carrier multiplex modulation categories of AM and FM.

Due to the added complexity of the multiplex systems, this section does not attempt to solve for the complete detected output signal as was done in previous sections. This section basically uses the large carrier-to-undesired-signal ratio approximations discussed in APPENDIX I and II, in order to calculate the output undesired signal power.

ANALOG OR VOICE MODULATION

The ratio of the desired signal to the undesired signal from the detector input to the low-pass filter output is to be determined. The undesired signal will consist of interference plus the system random noise. The basic approach will be that of a large carrier analysis where the analysis of the noise can approximately be treated independently of the signal and interference analysis.

A multiplex system in general consists of a carrier and subcarriers. A typical multiplex example is shown in Figure 7-1. This consists of N AM subchannels modulating an FM carrier.

Since multiplex systems consist of a carrier and subcarriers, the system's output ratio will be designated

$$R_{s-c} \qquad (7\text{-}1)$$

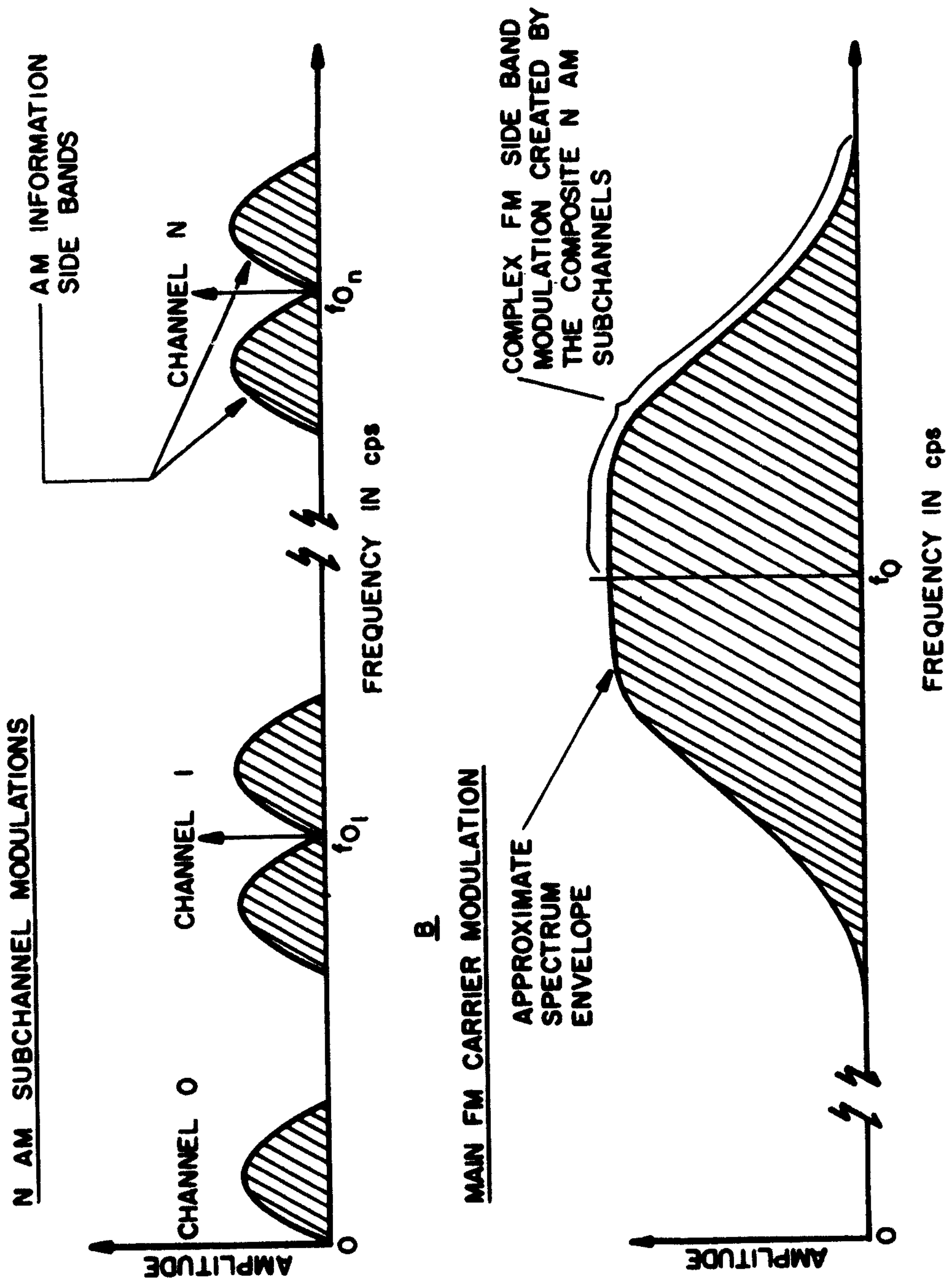


Figure 7-1. Spectral Representation of FM Multiplex

where S symbolizes the type of desired modulated signal, and C the type of carrier modulation. Two cases will now be discussed.

Case 1, $\underline{R_{AM-AM}}$. In order to simplify the method of obtaining the output ratio of the desired to undesired signal, the percent modulation of the subchannels will be assumed the same. The type of desired subchannel modulation will also be assumed to be of the same type and the same bandwidth per channel.* The general output interfering signal for this case will be assumed to be either AM or the modulation type of the desired carrier signal. This assumption eliminates the cross product terms of some of the desired-to-undesired signals and with this assumption neglects secondary interference terms while leaving the problem in a tractable nature. The form of the desired carrier and the undesired output signal for AM interference to an AM multiplex system is given by**

$$\frac{|V(t)|}{A_S} = m_S \cos \omega_S t + (m_I R_I{}^2/2) \cos \omega_I t + R_I \cos \Delta\omega t \qquad (7\text{-}2)$$

The desired input signal from which this relationship was derived was given as

$$S_I'(t) = A_S(1 + m_S \cos \omega_S t)\cos \omega_o t \qquad (7\text{-}3)$$

However, the actual form of the multiplex input is

$$S_I(t) = A_S[1 + \sum_i m_{Si}(1 + \cos \omega_S t)\cos \omega_{oi} t] \cos \omega_o t \qquad (7\text{-}4)$$

where

ω_{oi} = ith subcarrier frequency

m_{Si} = ith subcarrier modulation index.

The correct form of the output is therefore not that of equation (7-2) but one in which cross products between the subcarriers and

---

*Without these assumptions the detailed solution becomes considerably more involved although the basic method of solution is the same.

**The derivation of this equation was previously discussed in Section 4 and the block diagram and discussion up to equation (4-3) of that section apply here.

the off-tuned carrier exist. The resultant answer is difficult to completely describe; however, it still contains the off-tuned signal as the predominant interfering term. The approach here is therefore to assume the interfering signal of the form

$$I(t) = \sum_j m_{Ij}(R_I)^j \cos (j\Delta\omega t) + (m_I R_I{}^2/2)\cos \omega_I t \tag{7-5}$$

The undesired output of the ith channel is determined basically by the filter characteristics of this channel and the value of the off-tuning $\Delta\omega$. This situation is depicted for two possible interference situations in Figure 7-2. In the first case, $\Delta\omega$ is less than the bandwidth of a subchannel, hence multiple discrete interference is received in each subchannel. In the second case, $\Delta\omega$ is greater than a subchannel bandwidth and, hence, only a single representative tone interference is at most received in each channel.

The first idealized case is essentially the detector analysis problem of multiple CW signals into a linear detector. This problem was discussed in APPENDIX I and found difficult to handle. The only problem that can be reasonably handled is that of the limiting condition of noise. This problem will be covered as a special case of the second problem. The second problem is that of simple tone interference as previously discussed in SECTION 4. For this problem, the normalized signal into the second multiplex detector can be expressed as

$$\frac{|V(t)|}{A_S} = m_{Si}(1 + m_S \cos \omega_S t)\cos \omega_{oi} t + m_{Ij}'(R_I')^j \cos j\Delta\omega t \tag{7-6}$$

where

$$A_I' = A_I|H(j\Delta\omega')| = A_S R_I'|H(j\Delta\omega')|$$

m_{Ij}' = modified modulation coefficient due to $H(j\Delta\omega')$

$H(j\Delta\omega')$ = jth subchannel's filter characteristics at $j\Delta\omega'$. (Note: j is used here as a summation index.)

$$\Delta\omega' = |\omega_{oi} - j\Delta\omega|.$$

In this form the problem again reduces to that given in a previous discussion which referenced equation (I-71a).

With respect to this equation, two problems remain. These are the problems associated with assigning representative values

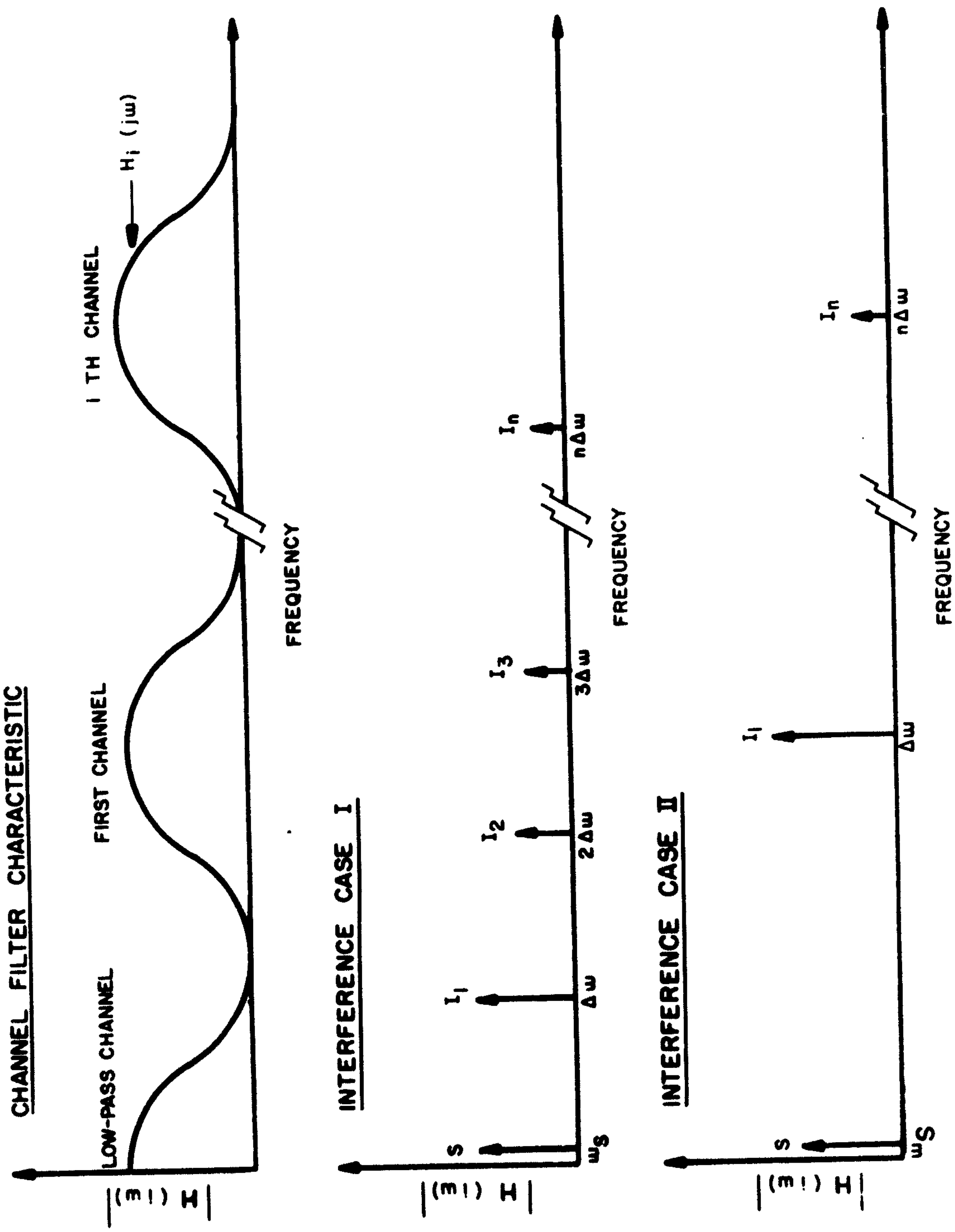


Figure 7-2. Representative Interference to an AM Multiplex System

to m_{Si} and m_S the modulation coefficients of the subchannel and the subchannel's intelligence signal. These values could basically be assigned on either a peak or rms basis.

Considering a peak basis if $m_S = 1$, the peak signal in each channel is $2m_{Si}A_S$. The maximum value of the composite signal is $2m_{Si}nA$. The maximum normalized value of m_{Si} is therefore

$$m_{Si} = \frac{1}{2n} \tag{7-7}$$

This type of restriction is overrestrictive. A more realistic criterion is to have the answer depend on an rms criterion. We therefore obtain in a straightforward fashion

$$[Am_{Si}(1 + m_S \cos \omega_S t)\cos \omega_{oi} t]_{RMS}$$

$$= \frac{Am_{Si}}{\sqrt{2}} [1 + 2m_S{}^2]^{1/2} \tag{7-8}$$

Since it is desired not to exceed a crest factor of 4* for the n channels on a peak voltage, basically we have that:

$$\frac{Am_{Si}}{\sqrt{2}}[1 + 2m_S{}^2]^{1/2} (n)^{1/2} \leq A/4 \tag{7-9}$$

Subsequently the maximum value of m_{Si} is found to be

$$m_{Si}{}^2 = \frac{1}{8n(1 + 2m_S{}^2)} \tag{7-10}$$

If $m_S = 1$, a worst case or peak type of value is obtained as

$$m_{Si}{}^2 = \frac{1}{24n} \tag{7-11}$$

If $m_S = 1/3$, a conservative value is obtained. This is a value that requires the 4:1 noise requirement not be exceeded for the

*See the discussion of the last paragraph on page 4-5.

total signal and consequently

$$m_{Si}^2 \approx \frac{1}{10n} \tag{7-12}$$

Since the second demodulation process is designed with a similar requirement on the subchannel, the second value, equation (7-12) will be used for analog or voice subchannels, when an average value approach is desired.

It is now desired to calculate the filtered desired output power. This is found from equations (7-6) and (7-10) to be

$$S_o = \frac{A_S^2}{8n} \frac{(m_S^2/2)}{(1 + 2m_S^2)} = \frac{A_S^2 m_S^2}{16n(1 + 2m_S^2)} \tag{7-13}$$

The filtered output interfering signal power is found to be

$$I_o = \frac{A_I''^2}{2} \tag{7-14}$$

where the " denotes the second filtering effect of the off-tuning. The output power ratio is subsequently found to be

$$(S/I)_o = \frac{m_S^2 R_S''^2}{8n(1 + 2m_S^2)} \tag{7-15}$$

The input signal power is found from equation (7-4) to be

$$S_I = \frac{A_S^2}{2} + \overline{(\sum_i A_S m_{Si})^2} \tag{7-16}$$

However, from equation (7-9) this is also equal to

$$S_I = \frac{A_S^2}{2} + \frac{A_S^2}{8} = \frac{5}{8} A_S^2 \tag{7-17}$$

The interference input power is equal to

$$I_I = \frac{A_I^2}{2} \tag{7-18}$$

The desired-to-undesired input-to-output ratio subsequently is found to be

$$R_{AM-AM} = \frac{(S/I)_I}{(S/I)_o} = \frac{5/4(A_S/A_I)^2 8n(1 + 2m_S^2)}{m_S^2(A_S/A_I'')^2} \tag{7-19a}$$

$$= 10n\left(\frac{1 + 2m_S^2}{m_S^2}\right)\left(\frac{A_I''}{A_I}\right)^2 \tag{7-19b}$$

when $A_S > A_I$.

The remaining problem is the calculation of the attenuation value A_I''. Although it is apparent that for any set of filter characteristics this can be obtained, a general formulation would be excessively involved, and hence will not be discussed. The most important factor of equation (7-19b) is that the ratio R varies directly with the number of channels n.

The previous type of ratio was obtained for the large carrier case, or that case in which the noise can be neglected. For the case in which the first order effects of noise are desired to be included, it is only necessary to calculate the output noise power and add this to the previous calculations.

The input white noise spectrum for an AM multiplex system is shown in Figure 7-3.

The input desired-to-undesired signal ratio can be obtained directly from this representation as

$$(S/I{+}N)_I = \frac{5/8A_S^2}{A_I^2/2 + N} = \frac{5A_S^2}{4(A_I^2 + 2N)} \tag{7-20}$$

where

N = the mean square input noise.

The output desired-to-undesired signal ratio is subsequently found to be

$$(S/I{+}N)_o = \frac{2A_S^2 m_S^2/16n(1 + 2m_S^2)}{(A_I''^2 + 2N/n)} \tag{7-21a}$$

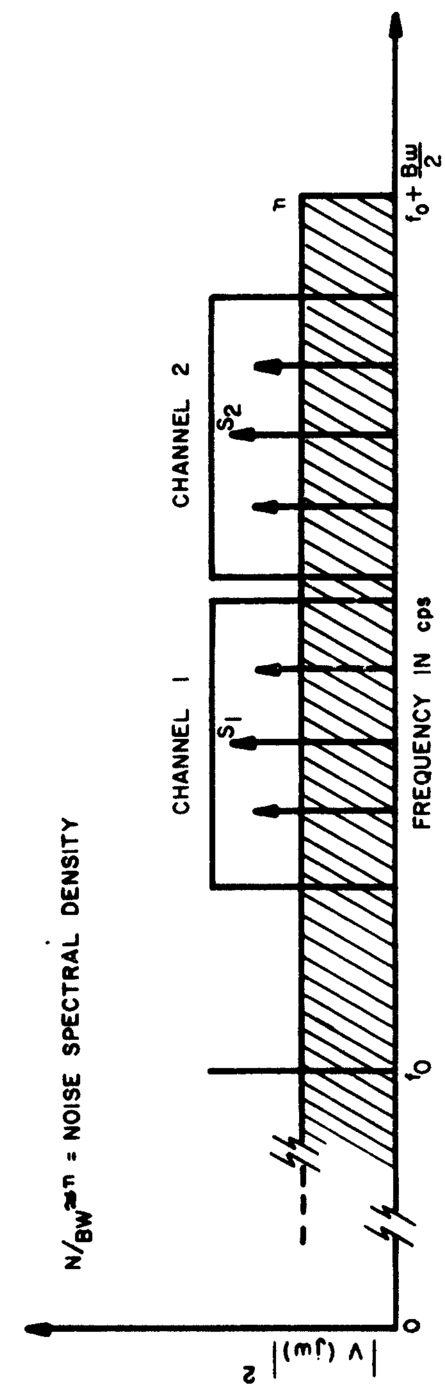


Figure 7-3. Input Noise Spectrum at Carrier Frequency f_o

$$= \frac{A_S{}^2 m_S{}^2}{8(1 + 2m_S{}^2)(nA_I{}''^2 + 2N)} \tag{7-21b}$$

The desired ratio is consequently

$$R_{AM-AM} = \frac{(S/I+N)_I}{(S/I+N)_o} = \frac{10(1 + 2m_S{}^2)(nA_I{}''^2 + 2N)}{m_S{}^2(A_I{}^2 + 2N)} \tag{7-22}$$

which for the case of N = 0 reduces to the previous result of equation (7-19b). The output ratio can therefore be obtained, within the limitations of this derivation, providing the desired signal's modulation coefficient, and the filtered and unfiltered interference to noise ratios are known.

Case 2, R_{AM-FM}. As the second multiplex problem, consider the case of the main carrier's modulation being FM. The desired input signal is therefore given by

$$S(t) = A_S \cos\left(\omega_o t + \sum_i \phi_i(t) + \theta_S\right) \tag{7-23}$$

where

$$\phi_i(t) = 2\pi D_i \int S_i(t)\,dt$$

$$S_i(t) = (1 + m_S \cos \omega_S t) \cos \omega_{oi} t$$

D_i = peak frequency deviation of the ith channel.

Consider the general FM interfering signal given by

$$I(t) = A_I(t) \cos[(\omega_o + \Delta\omega)t + \theta_I + \Phi_I(t)] \tag{7-24a}$$

$$= A_I(t) \cos[\Delta\omega t + \theta_I + \Phi_I(t) - \sum\phi_i(t)] \cos(\omega_o t + \sum\phi_i(t) + \theta_S)$$

$$- A_I(t) \sin[\Delta\omega t + \theta_I + \Phi_I(t) - \sum\phi_i(t)] \sin(\omega_o t + \sum\phi_i(t) + \theta_S) \tag{7-24b}$$

$$= X_I(t) \cos(\omega_o t + \sum\phi_i(t) + \theta_S) - Y_I(t) \sin(\omega_o t + \sum\phi_i(t) + \theta_S) \tag{7-24c}$$

The combined desired signal and interference is given by

$$V(t) = S(t) + I(t) = \sqrt{X_{S+I}{}^2 + Y_I{}^2}\, \cos\left[\omega_o t + \theta_I - \sum_i \phi_i(t) + \tan^{-1}\left(\frac{Y_I}{X_{I+S}}\right)\right] \tag{7-25}$$

where

$$\tan^{-1}\left(\frac{Y_I}{X_{I+S}}\right) = \tan^{-1}\left[\frac{A_I(t) \sin\,[\Delta\omega t + \theta_I + \phi_I(t) - \sum \phi_i(t)]}{A_S + A_I(t) \cos\,[\Delta\omega t + \theta_I + \phi_I(t) - \sum \phi_i(t)]}\right] \tag{7-26}$$

The output of the ideal balanced discrimination is

$$V_d(t)\Big|_{FM} = \frac{1}{2\pi}\left\{\frac{d}{dt} \sum_i \phi_i(t) + \frac{d}{dt}\left[\tan^{-1}\left(\frac{Y_I}{X_{I+S}}\right)\right]\right\} \tag{7-27a}$$

$$= \frac{1}{2\pi} \frac{d}{dt} \sum_i \phi_i(t) + I_o(t) \tag{7-27b}$$

For the representative case of tone modulation $I_o(t)$ reduces to equation (I-206)

$$I_o(t) = \sum_{n=1}^{\infty} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (-R_I)^n \left[\frac{rf_I}{n} - \frac{s\bar{f}_S}{n} + \Delta f\right] J_r(nB_I)$$

$$J_S(nB_S) \cos\,(n\Delta\omega t + n\theta_I + r\omega_I t - s\bar{\omega}_S t) \tag{7-28}$$

where

$\bar{f}_S$ is the representative average value of f_S.

Therefore, the interference is a function of all multiples of Δf, $\bar{f}_S$ and f_I. The desired signal consists of that shown in Figure 7-4. A particular channel is again selected and the filtered output of the desired and undesired signal obtained. The filtered output signal is however difficult to specify, due to Δf, $\bar{f}_S$ and f_I. The

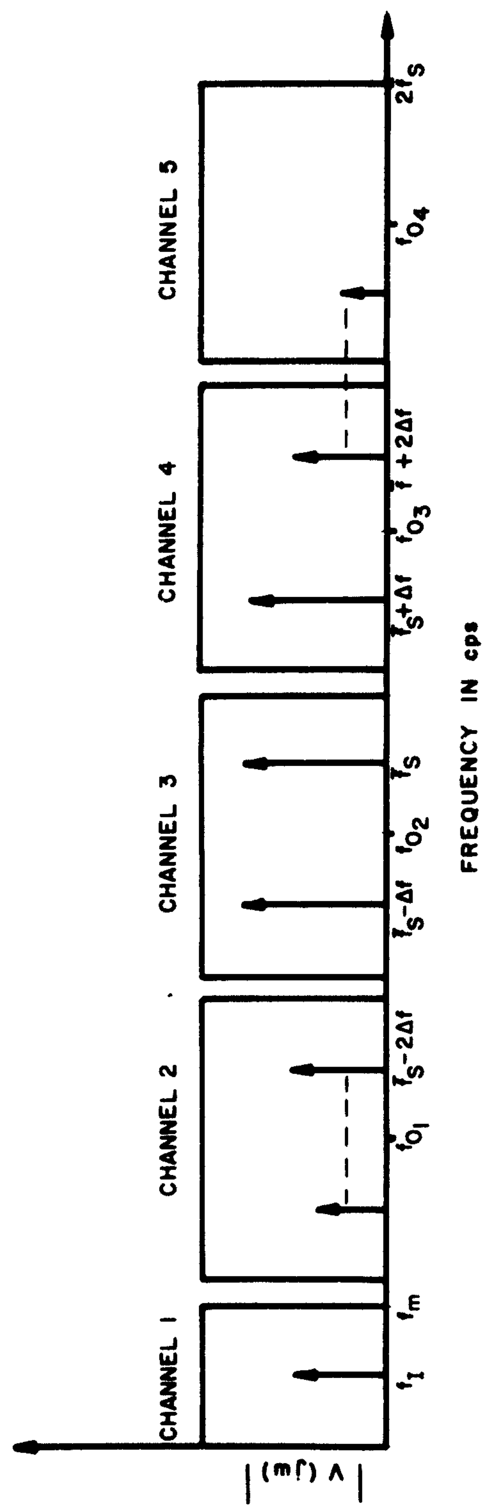


Figure 7-4. Representative Interference to an FM Multiplex System

value of $\overline{f}_S$ can be approximated by the bandwidth of the IF divided by four ($BW_{IF}/4$). The average frequency of f_I is a relatively small value compared to $\overline{f}_S$ for normal analog channels and so it is reasonable to approximate the previous interference signal by

$$I_o(t) = \sum_{n=1}^{\infty} \sum_{r=-\infty}^{\infty} (-R_I)^n [n\Delta f + r\overline{f}_S] J_o(r\overline{f}_S) \cos [n\Delta\omega t + 2\pi r\overline{f}_S t] \tag{7-29}$$

Although the actual details have been considerably simplified, the evaluation of the filtered output interference signal remains involved, due to the wide range of values possible for Δf. The problem, therefore, again becomes that of multiple tone interference against an AM system. This problem is discussed in APPENDIX I and was found difficult to handle. Because of this difficulty, a single representative tone proportional to the frequency ($\Delta f + \overline{f}_S$) will be assumed as the interfering signal. For this case the problem reduces to the type previously discussed in case 1 of this section and in SECTION 4.

The undesired noise output from the ith channel can be approximately computed by using the basic equation (II-22) from APPENDIX II, and realizing that the approximate output power density is

$$n_{0i} \approx \frac{2n}{A_S^2} f_{0i}^2 \tag{7-30}$$

where

$$n = (N/BW_{IF}) = \text{Noise power per cycle} \tag{7-31}$$

The total desired-to-undesired power ratio for the ith channel can now be obtained from equations (7-23), (7-29) and (7-30). The output is obtained as

$$(S/N+I)_o = \frac{D_i^2}{(8nf_m f_{0i}^2/A_S^2) + R_I'^2(\Delta f + \overline{f}_S)^2} \tag{7-32}$$

The input desired-to-undesired ratio is readily obtained as

$$(S/N+I)_I = \frac{A_S{}^2}{A_I{}^2\overline{A_I{}^2(t)} + 2nBW_{IF}} \qquad (7\text{-}33)$$

where

$\overline{A_I{}^2(t)}$ = the normalized mean square interference modulation

The desired ratio is therefore

$$R_{AM-FM} = \frac{\left[8nf_m + A_I{}'^2\left(\frac{\Delta f + \overline{f}_S}{f_{0i}{}^2}\right)\right]^2}{\left[A_I{}^2\overline{A_I{}^2(t)} + 2N\right]} \left(\frac{f_{0i}}{D_i}\right)^2 \qquad (7\text{-}34)$$

In ordinary multiplex design the ratio f_{0i}/D_i is kept a constant. Again, a derivation could be based upon a peak or rms basis. Using an rms basis the derivation is found to be

$$D_{RMS} = \left[\sum_i^n 3/4\ D_i{}^2\right]^{1/2} = \sqrt{3/2}\left(\frac{D_i}{f_{0i}}\right)(BW_i)[1^2 + 3^2 + \ldots (2n-1)^2]^{1/2} \qquad (7\text{-}35a)$$

$$= \sqrt{3/2}\ \frac{D_i}{f_{0i}}\ (BW_i)[(4n^3 - n)/3]^{1/2} \qquad (7\text{-}35b)$$

$$\approx \left(\frac{D_i}{f_{0i}}\right)(BW_i)\ n^{3/2} \qquad (7\text{-}35c)$$

where

BW_i = bandwidth of the ith multiplex channel

It is also true that, approximately,

$$BW_i \approx (BW_{IF})/(2n-1) \qquad (7\text{-}36)$$

Combining equations (7-31), (7-34), (7-35) and (7-36) and using an average to peak crest factor of four, we obtain*

$$R_{AM-FM} = \frac{64n^3}{(2n-1)^2} \frac{\left[\frac{4N}{(2n-1)} + A_I'^2\left(\frac{\Delta f + \overline{f}_S}{f_{0i}}\right)^2\right]}{2N + A_I^2\overline{A_I^2(t)}} \qquad (7\text{-}37)$$

When the number of multiplex channels is much greater than one (i.e., when 2n >> -1) the maximum ratio R_{AM-FM} is obtained for

$$\Delta f = (2n-1)f_m$$

$$f_{0i} = \frac{(2n-1)f_m}{2}$$

Equation (7-37) then reduces to

$$R_{AM-FM} = 16n\frac{\left[\frac{2N}{n} + 36A_I'^2n^2\right]}{\left[2N + A_I^2\overline{A_I^2(t)}\right]} \qquad (7\text{-}38)$$

The output ratio is, therefore, again a direct function of the number of multiplex channels and requires the values of the filtered and unfiltered interference-to-noise ratios for evaluation.

DISCUSSION OF RESULTS

A basic difference between the derivation in this case (AM-FM) and the previous case (AM-AM) is the manner in which the FM demodulation spread the interfering signal throughout the band. In the AM-AM problem the main interfering signal was only demodulated to one subchannel. In the AM-FM the main FM demodulation process spread the interfering signal to more subchannels (but probably not all) and hence caused a degradation in a more widespread manner than in the AM case. In both cases the ratio of the input to output signal to interference ratio is a direct function of the number of multiplex channels.

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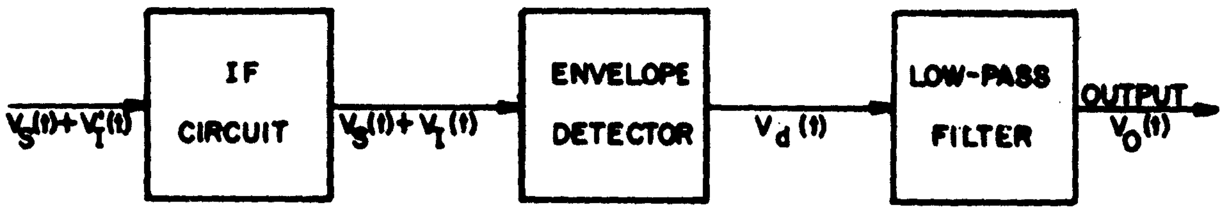
*Note that from the previous derivation f_{0i} goes to $BW_1/4$ for the 0th channel.

SECTION 8

A DESIRED AM SIGNAL INTERFERED WITH BY AN UNDESIRED PULSED SIGNAL

INTRODUCTION

The portion of the amplitude modulated (AM) receiver to be treated in the following analysis is depicted in Figure 8-1.



$v_S(t)$ = Desired signal

$v_I'(t)$ = IF input undesired signal

$v_I(t)$ = IF output undesired signal

$v_d(t)$ = Detector output signal

$v_o(t)$ = Low-pass filter output signal.

Figure 8-1. AM Receiver, Portion Analyzed

The assumptions used for the following receiver subsections and the input signals can be stated as follows:

1. The input desired signal is given by

$$v_S(t) = A_S[1 + m_S(t)] \cos \omega_o t \tag{8-1}$$

where the only constraint on the modulating waveform is

$$\overline{m_S^2(t)} \leq 0.1 \tag{8-2}$$

2. System noise is assumed small in comparison with the desired signal and is consequently neglected.

3. The input interference signal is given by

$$v_I'(t) = A_I' m_I'(t) \cos\left[(\omega_0 + \Delta\omega)t + \theta\right] \tag{8-3}$$

where $m_I(t)$ is a binary on-off waveform of unit amplitude. It is assumed that the average width of the pulses of $m_I(t)$ obeys the inequality

$$\beta_{IF} \geq 1 \tag{8-4}$$

where β_{IF} is the IF half-bandwidth.

No restrictions are placed upon the values of A_I or $\Delta\omega$ within the assumptions of APPENDIX III.

4. The IF amplifier is assumed to pass $v_S(t)$ without modification and to alter $v_I'(t)$ by an amplitude factor $g(\Delta f)$ which depends upon the IF selectivity curve (where $g(0) = 1$). Hence, the interference signal at the IF output is

$$v_I(t) = g(\Delta f)\, v_I'(t) = A_I m_I(t) \cos\left[(\omega_0 + \Delta\omega)t + \theta\right] \tag{8-5}$$

where $$A_I \equiv A_I' \, g(\Delta f) \tag{8-6}$$

This analysis will treat A_I as the effective amplitude of the input interference.

The transformation from $m_I'(t)$ to $m_I(t)$ is discussed in APPENDIX III.* Since this transformation is a function of the particular IF transfer function and Δf that is chosen, it will

*Note the change in the definition of m due to the change from a continuous modulation system to a pulse system.

be sufficient for this discussion to leave the transformation symbolically as m(t). For the particular case of a symmetrical transfer function, they are equal. See equation (III-6) in APPENDIX III. Also, the duty cycle of the pulse interference is assumed to be

$$\delta[\equiv \overline{m_I(t)}] \approx 0.5 \tag{8-7}$$

5. The envelope detector is assumed to be ideal.

6. The low-pass filter is a conventional RC circuit whose transfer function is given by

$$H(jf) = \frac{1}{1 + j(f/\beta)} \tag{8-8}$$

where β is the audio bandwidth of the modulating waveform $m_S(f)$.

DERIVATION OF THE DETECTOR OUTPUT SIGNAL

Since $m_I(t)$ is a binary on-off signal of amplitude 1, we can express the detector output in the approximate form

$$V_d(t) = V_{d0}(t)[1 - m_I(t)] + V_{d1}(t)[m_I(t)] \tag{8-9}$$

where

$V_{d0}(t)$ is the detector output for times at which no interference is present ($m_I = 0$),
and $V_{d1}(t)$ is the detector output for times at which interference is present ($m_I = 1$).

This expression neglects the detector cross product terms due to transients of $m_I(t)$ from the assumption of the final value states $m_I = 0$ and $m_I = 1$. It is only strictly true in the limit of a large or small signal-to-interference ratio as can be seen in equation (I-85). The detector input signal is given by

$$v_S(t) + v_I(t) = A_S[1 + m_S(t)] \cos \omega_o t + A_I m_I(t) \cos [(\omega_o + \Delta\omega)t + \theta] \tag{8-10}$$

we can find $v_{d0}(t)$ and $v_{d1}(t)$ by determining the envelope of equation (8-10) for the cases $m_I = 0$ and $m_I = 1$, respectively. When $m_I = 0$, the envelope of the detector input is clearly

$$V_{d0}(t) = A_S[1 + m_S(t)] \qquad (8\text{-}11)$$

while, for $m_I = 1$, the envelope of the detector input can be shown to be*

$$V_{d1}(t) = [A_S{}^2(t) + A_I{}^2 + 2A_S A_I \cos\phi]^{1/2} \qquad (8\text{-}12)$$

where

$$A_S(t) \equiv A_S[1 + m_S(t)] \qquad [=V_{d0}(t)] \qquad (8\text{-}13)$$

and

$$\phi \equiv \Delta\omega t + \theta \qquad (8\text{-}14)$$

The objective now is to express $V_{d1}(t)$ in a form suitable for assessing the output interference.

By adding and subtracting $2A_S(t)A_I$ inside the brackets of equation (8-12), the expression for $V_{d1}(t)$ can be altered to the form

$$V_{d1}(t) = \{[A_S{}^2(t) + 2A_S(t)A_I + A_I{}^2] - 2A_S(t)A_I(1 - \cos\phi)\}^{1/2}$$

$$= [A_S(t) + A_I]\left\{1 - \frac{4R_S(t)}{[1 + R_S(t)]^2} \sin^2\frac{\phi}{2}\right\}^{1/2} \qquad (8\text{-}15)$$

where

$$R_S(t) = \frac{A_S(t)}{A_I} \qquad (8\text{-}16)$$

It is important to note that

$$0 \leq \frac{4R_S(t)}{[1 + R_S(t)]^2} \leq 1$$

*See (I-20) for the basic formulation.

for all possible values of $R_S(t)$ and ϕ. Hence, we can expand the square-root term of equation (8-15) into a series. The first three terms of the series are usually the most important terms for interference analysis. These terms can be obtained from equations (I-101), (I-102) and (I-103) of APPENDIX I as

$$A_0 = \frac{1}{1 + R_S(t)} - \sum_{n=2}^{\infty} \frac{(2n-3)!\,(2n)!}{4^{n-1}(n!)^3(n-2)!} \left[\frac{R_S{}^n(t)}{[1 + R_S(t)]^{2n-1}}\right] \tag{8-17}$$

$$A_1 = \frac{R_S(t)}{1 + R_S(t)} + \sum_{n=2}^{\infty} \frac{(2n-3)!\,(2n)!}{4^{2n-1}(n!)^3(n-2)!} \frac{2n}{n+1} \left[\frac{R_S{}^n(t)}{[1 + R_S(t)]^{2n-1}}\right] \tag{8-18}$$

$$A_2 = -\sum \frac{(2n-3)!\,(2n)!}{4^{2n-1}(n!)^3(n-2)!} \frac{2n}{n+1} \frac{n-1}{n+2} \left[\frac{R_S{}^n(t)}{[1 + R_S(t)]^{2n-1}}\right] \tag{8-19}$$

Using equation (8-14) we can now approximate the output interference signal of the detector by

$$A_I(t) \approx A_I m_I(t)[A_0 + A_1 \cos(\Delta\omega t + \theta) + A_2 \cos(2\Delta\omega t + 2\theta)] \tag{8-20}$$

where A_0, A_1, and A_2 are given in terms of $R_S(t)$ by equation (8-17), (8-18), and (8-19). The main approximation involved in this expression is the exclusion of the higher order terms of the series. This is justified by the increasingly reduced values of $A_P[R_S(t)]$ for all $R_S(t)$ as P increases beyond 2, and also by the fact that these terms are generally suppressed by the low-pass filter.

We are now in a position to discuss the most important approximation in this development. It is apparent from the fact that $R_S(t) = (A_S/A_I)[1 + m_S(t)]$ that A_0, A_1, and A_2 are complicated

functions of $m_S(t)$ and, therefore, of time. Thus, for example, $A_1 \cos(\Delta\omega t + \theta)$ can be represented by a spectral distribution about Δf, where the sideband distribution is directly related to $m_S(t)$. We can therefore write A_1 in the general form

$$A_1(t) = A_{C1}[1 + f_1\{m_S(t)\}] \tag{8-21}$$

where $f_1(0) = 0$, and A_{C1} is a function of (A_S/A_I) and the time moments of $m_S(t)$, i.e., $\overline{m_S(t)}$, $\overline{m_S{}^2(t)}$, etc. Similar results apply to $A_0(t)$ and $A_2(t)$ as well. In the special case where $m_S(t) = 0$, (no modulation) we have $A_1(t) = A_{C1}$, i.e., a time-invariant term whose magnitude is a function solely of (A_S/A_I). Whereas $m_S(t)$ is in fact not identically zero, it has been postulated that most of the signal energy is contained in the carrier, and that less than 10% of its total energy is contained in the spectral components due to $m_S(t)$. Hence, the approach is to neglect the components of $A_1(t)$ due to $m_S(t)$, that is, to represent A_1 as a nominally constant term, obtained by replacing A_S $\{\equiv A_S[1 + m_S(t)]\}$ in equation (8-18) by a constant voltage having the same rms value as $A_S(t)$. Thus

$$A_1 \simeq A_1\{R_S(t)\}\Big|_{R_S(t)} \tag{8-22}$$

when

$$R_S(t) = \frac{A_S}{A_I}\,[1 + \overline{m_S{}^2(t)}]^{1/2}$$

and similarly for A_0 and A_2. This is equivalent to assuming that all of the sideband energy of the desired signal has been "compressed" so as to reside entirely at the carrier frequency.

It can be argued that the consequences of the above assumption are negligible for the case of the terms centered about Δf and $2\Delta f$. Thus, since the actual sideband power distributed about Δf is quite small compared to the actual carrier power, a single carrier representation of this signal is quite adequate

as long as the carrier amplitude, equation (8-22), truly reflects the total signal power. A similar representation may be made for the signal centered about $2\Delta f$ and the approximation for A_2. For the signal centered at zero frequency, however, mathematically condensing all of the signal power into the dc term can be misleading, because this term is removed from the output by dc blocking. Thus, all of the ac interference power distributed about zero frequency has seemingly been eliminated by our mathematical approximation for A_0. However, it can be demonstrated that, for the case considered here, this approximation will not seriously affect our results. This topic is discussed in APPENDIX I, equation (I-39).

To summarize the above discussion: we have represented the detector output interference signal by equation (8-20), where A_0, A_1 and A_2 are derived from equations (8-18), (8-19), and (8-20) for $R_S(t)$ having the value (S_{rms}/A_I). This signal is then processed through the low-pass filter to the receiver output. In the next section, we will present the results of this in a form which should be useful in subsequent system performance analysis problems.

From the above discussion, we conclude that the detector output can be expressed in the form

$$V_d(t) = \underbrace{\underbrace{A_S}_{\substack{\text{DC}\\ \text{Term}}} + \underbrace{A_S m_S(t)}_{\substack{\text{Modulation}\\ \text{Term}}}}_{\text{Desired Signal}}$$

$$\left.\begin{aligned} &+ \underbrace{\delta A_I A_o}_{\substack{\text{DC}\\ \text{Term}}} + \underbrace{\sqrt{\delta(1-\delta)}\, A_I A_o a(t)}_{\substack{\text{AC Term about}\\ f\,=\,0}} + \underbrace{A_I A_1 m_I(t) \cos(\Delta\omega t + \theta)}_{\substack{\text{AC Term about}\\ f\,=\,\Delta f}} \\ &+ \underbrace{A_I A_2 m_I(t) \cos(2\Delta\omega t + 2\theta)}_{\substack{\text{AC Term about}\\ f\,=\,2\Delta f}} \end{aligned}\right\} \text{Interference Signal} \tag{8-23}$$

Thus, the major components of the interference signal are (1) dc voltage, over and above that due to the desired signals, which may affect the action of the AGC circuit; and (2) three distinct ac pulse signals with carrier frequencies of 0, Δf, and $2\Delta f$. A useful representation of the interference may therefore be one which gives the value of the dc term and the receiver output power due to each of the ac terms, as functions of (S_{rms}/A_I), δ, and $(\Delta f/\beta)$. From equation (8-23) we see that the dc term is given simply by

$$\Delta V_{DC} = \delta A_I A_o \tag{8-24}$$

In order to find the receiver output power due to the ac terms of the interference signal, we will invoke one of our earlier assumptions, namely, that the average width of the interference pulses is no smaller than $1/\beta_{IF}$, where β_{IF} is the IF half-bandwidth and nominally equal to the low-pass filter bandwidth. Hence, as each ac interference term of equation (8-23) is processed through the low-pass filter, its waveshape will be modified only slightly, the major change in its form being an amplitude reduction due to the steady-state filter response at the center frequency. That is, we can approximate the receiver output due to the ac interference terms by means of the following expression (where this is consistent with the original IF off-tuning assumptions),

$$A_I(t)_o = A_I A_o \sqrt{\delta(1-\delta)}\, a(t) + \frac{A_I A_1 m_I(t)}{\sqrt{1 + (\Delta f/\beta)^2}} \cos(\Delta\omega t + \theta + \phi_1) + \frac{A_I A_2 m_I(t)}{\sqrt{1 + (2\Delta f/\beta)^2}} \cos(2\Delta\omega t + 2\theta + \phi_2) \tag{8-25}$$

where ϕ_1 and ϕ_2 are the low-pass filter phase shifts at frequencies Δf and $2\Delta f$, respectively. The total power in each of these terms is now found as follows:

1. The power contained in the component centered at zero frequency is readily found to be

$$P_{IO} = \delta(1 - \delta)(A_I A_o)^2\, \overline{a^2(t)} = \delta(1 - \delta)(A_I A_o)^2 \tag{8-26}$$

2. The power contained in the component whose carrier is Δf is similarly found to be

$$P_{I1} = \frac{(A_I A_1)^2}{1 + (\Delta f/\beta)^2} \overline{\{m_I^2(t) \cos^2 (\Delta\omega t + \theta + \phi_1)\}}$$

$$= \frac{1}{2} \frac{\delta (A_I A_1)^2}{1 + (\Delta f/\beta)^2} \tag{8-27}$$

3. The power contained in the component whose carrier is $2\Delta f$ is also found to be

$$P_{I2} = \frac{1}{2} \frac{\delta (A_I A_2)^2}{[1 + (2\Delta f/\beta)^2]} \tag{8-28}$$

The latter two results presuppose that the pulses of $m_I(t)$ do not occur at a periodic rate which is an integral submultiple of Δf. This assumption holds for all but the most coincidental (and therefore trivial) cases.

If P_{I0} were derived rigorously, i.e., without the approximation implied by equation (8-25), the result obtained would differ from equation (8-26) by at most -0.9 db, this discrepancy occurring for the limiting condition where the spectrum of a(t) is uniformly distributed over $-\beta \leq f \leq +\beta$. If P_{I1} and P_{I2} were derived rigorously, then for $\delta = 0.5$ the results obtained would differ from equations (8-27) and (8-28) by at most -0.45 db, this discrepancy occurring in the extreme case where $\Delta f/\beta \ll 1$ and the spectrum of a(t) is uniformly distributed in the range $0 \leq f \leq \beta$.

The pertinent features of the output interference are thus seen to be characterized by V_{dc}, P_{I0}, P_{I1}, and P_{I2}, respectively. The expressions for these terms are simple functions of δ (which is assumed to be in the neighborhood of 0.5), $(\Delta f/\beta)$, and the coefficients A_o, A_1, and A_2.

For purposes of computation, the expressions for A_o, A_1, and A_2 can be approximated by finite sums, where the number of terms required to achieve a given accuracy depends on the value of $R_S(t)$. Unfortunately, as $R_S(t)$ approaches unity, the required

number of terms becomes unreasonably large. This problem can be circumvented, however, by considering equation (8-15), for the particular case when $A_S(t) = A_I$. We then find that

$$\begin{aligned} V_{d1}(t) &= [A_S(t) + A_I][1 - \sin^2 \tfrac{\phi}{2}]^{1/2} \\ &= [A_S(t) + A_I]\left|\cos \tfrac{\phi}{2}\right| \qquad (8\text{-}29) \\ &= [A_S(t) + A_I]\left[\tfrac{2}{\pi} + \tfrac{4}{3\pi}\cos\phi - \tfrac{4}{15\pi}\cos 2\phi + \ldots\right] \end{aligned}$$

But, since $A_S(t) = A_I$, we can after some manipulation write this as

$$V_{d1}(t) \approx A_S(t) + A_I[.27 + .85 \cos\phi - .17 \cos 2\phi + \ldots] \qquad (8\text{-}30)$$

Hence, the true values of A_o, A_1, and A_2 for the limiting case $R_S(t) = 1$ are 0.27, 0.85, and -0.17 respectively. Using these results when $R_S(t) = 1$, and computing equations (8-17), (8-18), and (8-19) over values of n up to 18 when $R_S(t)$ is not close to unity, we have obtained the curves of A_o, A_1, and A_2 shown in Figure 8-2.

Using the low-pass filter function

$$|H(f)|^2 = \frac{1}{1 + (\Delta f/\beta)^2} \qquad (8\text{-}31)$$

and equations (8-27) and (8-28), P_{I1} and P_{I2} can readily be obtained from the values of A_1 and A_2 given in Figure 8-2.

DISCUSSION OF RESULTS

The most important restrictions placed upon the pulse interference analysis for the AM receiver have been the assumptions that

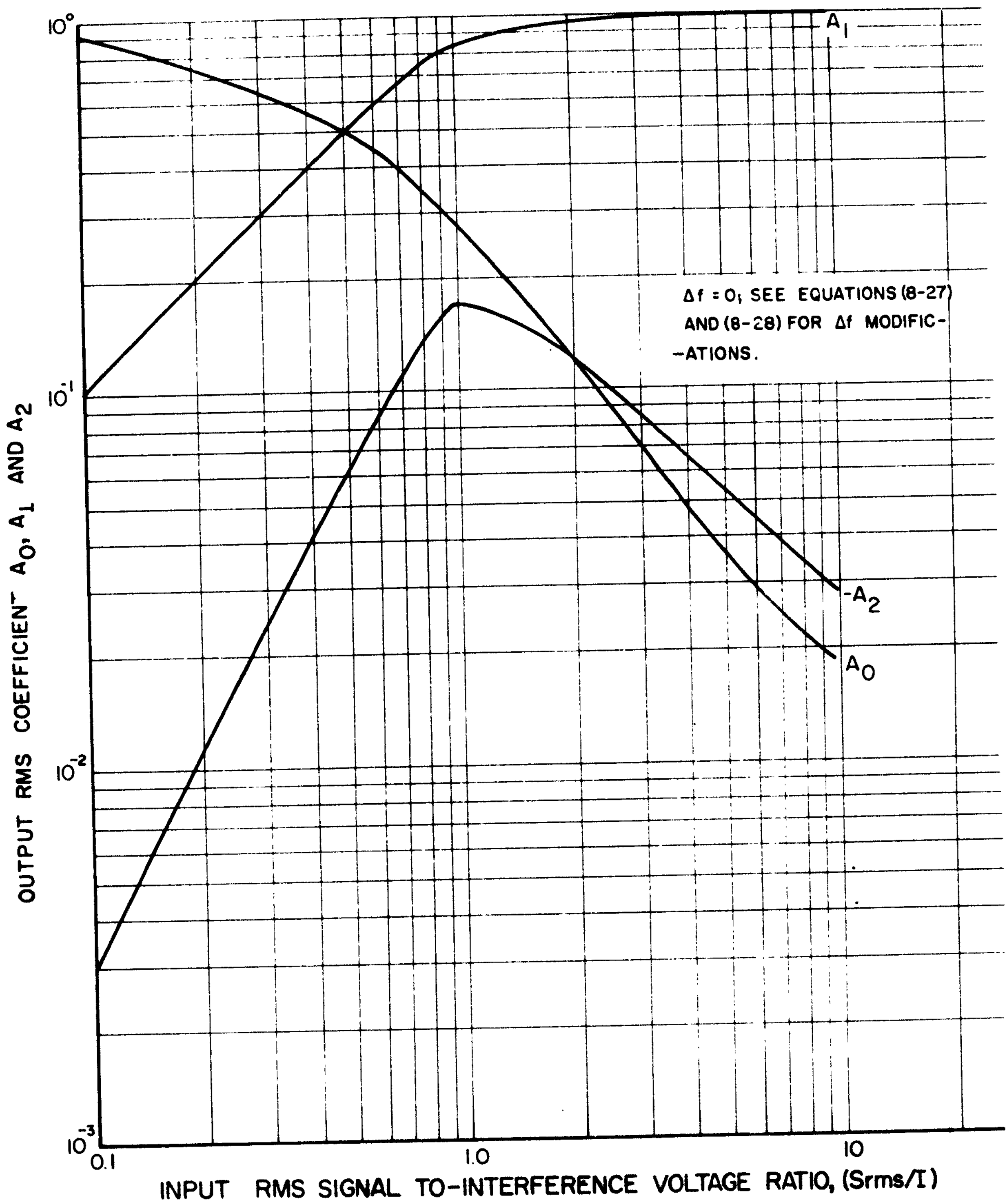


Figure 8-2. Normalized Carrier Amplitudes for Output Interference Due to Interaction of Input Pulse Interference with Desired Signal $[\overline{m_S^2(t)} \le 0.1]$

1. $m_S^2(t) \leq 0.1$

2. $\delta \approx 0.5$

3. $\beta t \geq 1$.

The first condition permits us to neglect, in the receiver output interference signal, the sideband distributions about Δf and $2\Delta f$ due to $m_S(t)$. This in turn also validates neglecting the secondary cross product terms due to the transients of the pulsed interfering signal. The second condition, in combination with the first, permits us to do the same with respect to the ac interference component centered about zero frequency. The third condition permits us to treat the effects of the low-pass filter in a mathematically simple way, and tacitly, to do the same with respect to the IF amplifier. This last condition is probably the least vital in simplifying the analytical approach, and could be violated at the expense of some added but containable mathematical complexity. The first two conditions, aside from permitting both a convenient solution of the problem and a useful representation of the results thereof, are important for two other reasons:

1. They correspond respectively to the most common type of AM system, and the most common type of pulse interference encountered by such a system.
2. They provide, individually and collectively, a worst-case estimate of the effects of pulse interference, that is, the interference effects observed when the modulation power of the desired signal is low relative to the carrier power, and when the interference pulses occur for a large fraction of time.

The basic problem remaining for degradation analysis is to then obtain performance curves for undesired signals of approximately a 50% duty cycle.

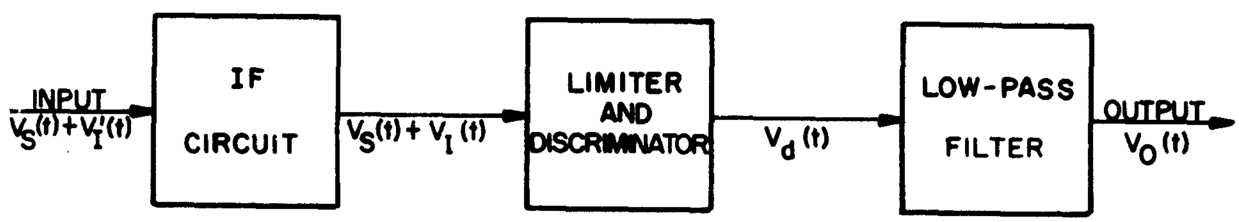
The results of this section generally apply to both analog and voice modulation.

SECTION 9

A DESIRED FM SIGNAL INTERFERED WITH BY AN UNDESIRED PULSED SIGNAL

INTRODUCTION

The portion of the frequency modulated (FM) receiver to be treated in the following analysis is depicted in Figure 9-1.



$v_S(t)$ = Desired signal

$v_I'(t)$ = IF input undesired signal

$v_I(t)$ = IF output undesired signal

$v_d(t)$ = Detector output signal

$v_o(t)$ = Low-pass filter output signal.

Figure 9-1. FM Receiver, Portion Analyzed

The assumptions used for the following receiver subsections and the input signals can be stated as follows:

1. The input desired signal is given by

$$v_S(t) = A_S \cos [\omega_o t + \phi_S(t)] \tag{9-1}$$

where

$$S_K(t) \equiv \frac{1}{2\pi} \frac{d\phi_S(t)}{dt} \tag{9-2}$$

is the desired modulating waveform. We are considering only the wideband type of FM system so that the bandwidth in cps of $v_S(t)$ is twice the maximum deviations of $S_K(t)$, i.e.,

$$\text{Bandwidth of } v_S(t) \equiv 2B \doteq \frac{1}{\pi}\omega_{Smax} \tag{9-3}$$

The bandwidth of the modulating signal $S_K(t)$ is β, where (B/β) is greater than one. (For purposes of illustration, we will assume later on that $B/\beta = 5$.)

2. System noise is assumed small in comparison with the desired signal and is consequently neglected.

3. The input interference signal is given by

$$v_I'(t) = A_I'm_I'(t) \cos [(\omega_o + \Delta\omega)t + \theta] \tag{9-4}$$

where $m_I'(t)$ is a binary on-off waveform of unit amplitude. It is assumed that the average width of the pulses of $m_I'(t)$ obeys the inequality

$$\beta_{IF}\tau \geq 1 \tag{9-5}$$

where β_{IF} is the IF half-bandwidth. No restrictions are placed upon the values of I' or $\Delta\omega$ within the assumptions of APPENDIX III.

4. The IF amplifier is assumed to pass $v_S(t)$ without modification, and to alter $v_I'(t)$ by an amplitude factor $g(\Delta f)$ which depends upon the IF selectivity curve (where $g(0) = 1$). Hence, the interference signal at the IF output is

$$v_I(t) = g(\Delta f)v_I'(t) = A_I m_I(t) \cos [(\omega_o + \Delta\omega)t + \theta)] \tag{9-6}$$

where

$$A_I \equiv A_I'g(\Delta f) \tag{9-7}$$

The transformation from $m_I'(t)$ to $m_I(t)$ is discussed in APPENDIX III. Since this transformation is a function of the particular IF transfer function and Δf that is chosen, it will be sufficient for this discussion to leave the transformation symbolically as m(t). For the particular case of a symmetrical transfer function they are of the same form.

Also, the duty cycle of the pulse interference is assumed to be

$$\delta[\equiv \overline{m_I(t)}] \approx 0.5 \tag{9-8}$$

The analysis will treat A_I as the effective amplitude of the input interference.

5. We assume an ideal limiter and discriminator, so that $v_d(t)$ is proportional to the instantaneous frequency deviation of the IF output signal from f_o.

6. The FM system is assumed to use pre-emphasis, so that the low-pass filter transfer function contains a simple corner at the corresponding de-emphasis frequency, f_d. In addition, it is assumed to contain a simple corner at the upper limit, β, of the modulating signal spectrum. Thus, we assume a transfer function of the form

$$H(jf) = \frac{1}{[1 + (jf/f_d)][1 + (jf/\beta)]} \tag{9-9}$$

For purposes of illustration, we will assume later on that $(\beta/f_d) = 7.07$.

DERIVATION OF THE DISCRIMINATOR OUTPUT SIGNAL

The discriminator output can be written in the form

$$V_d(t) = K_{FM}[f_{inst}(t) - f_o] \tag{9-10}$$

where $f_{inst}(t)$ is the instantaneous frequency of the IF output and K_{FM} is the FM detector constant. Since $m_I(t)$ is a binary on-off signal of amplitude 1, we can express the discriminator output in the general form

$$V_d(t) = V_{d0}(t)[1 - m_I(t)] + V_{d1}(t)[m_I(t)] \tag{9-11}$$

where $V_{d0}(t)$ is the discriminator output for times at which no interference is present ($m_I = 0$),

and $V_{d1}(t)$ is the discriminator output for times at which interference is present ($m_I = 1$).

This formulation neglects the detector cross product terms due to transients of $m_I(t)$. This stems from the assumption of the final value which states $m_I = 0$ and $m_I = 1$. It is only strictly true in the limit of a large or small signal-to-interference ratio as can be seen in equation (I-85), APPENDIX I.

Since the IF output signal is given by

$$v_S(t) + v_I(t) = A_S \cos[\omega_o t + \phi_S(t)] + A_I m_I(t) \cos[(\omega_o + \Delta\omega)t + \theta] \tag{9-12}$$

we can find $V_{d0}(t)$ and $V_{d1}(t)$ by determining the instantaneous frequency of equation (9-12) for the cases $m_I = 0$ and $m_I = 1$, respectively. When $m_I = 0$, the IF output is a single phasor whose instantaneous phase is

$$\phi_{inst}(t) = \omega_o t + \phi_S(t) \tag{9-13}$$

Hence, the instantaneous frequency is

$$f_{inst}(t) \equiv \frac{1}{2\pi}\frac{d\phi_{inst}(t)}{dt} = f_o + \frac{1}{2\pi}\frac{d\phi_S(t)}{dt} \tag{9-14}$$

and we obtain

$$V_{d0}(t) = K_{FM}\frac{d\phi_S(t)}{dt} = K_{FM}S_K(t) \tag{9-15}$$

When $m_I = 1$, the IF output is the sum of two phasors as in Figure 9-2.

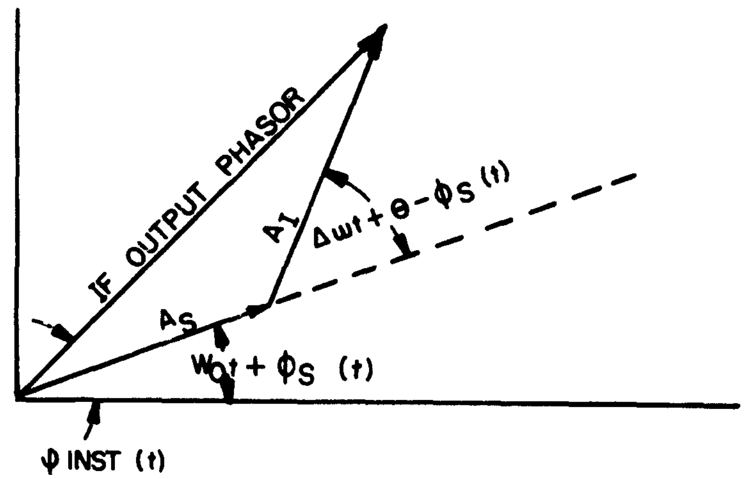


Figure 9-2. IF Output Phasors

The instantaneous phase of the combination shown in Figure 9-2 can be obtained from equation (I-141b), setting $I_K(t) = 0$, as

$$\phi_{inst}(t) = \omega_o t + \phi_S(t) + \tan^{-1} \frac{A_I \cos \Delta}{A_S + A_I \sin \Delta} \qquad (9\text{-}16)$$

where

$$\Delta \equiv \Delta\omega t + \theta - \phi_S(t) \qquad (9\text{-}17)$$

(For convenience, and with no loss in generality, we will assume $\theta =$, from here on.) The instantaneous frequency can now be found to be

$$f_{inst}(t) = f_o + S_K(t) + \frac{1}{2\pi} \frac{d\Delta}{dt} \left(\frac{1 + R_S \cos \Delta}{1 + R_S{}^2 + 2R_S \cos \Delta} \right) \qquad (9\text{-}18)$$

where

$$R_S \equiv A_S / A_I \qquad (9\text{-}19)$$

and

$$\frac{1}{2\pi} \frac{d\Delta}{dt} = \Delta f - S_K(t) \qquad (9\text{-}20)$$

The discriminator output, when $m_I = 1$, is therefore

$$V_{d1}(t) = K_{FM}\left[S_K(t) + \frac{(1 + R_S\cos\Delta)[\Delta f - S_K(t)]}{1 + R_S^2 + 2R_S\cos\Delta}\right] \tag{9-21}$$

and in general, $V_d(t)$ can be expressed in the form

$$V_d(t) = S_o(t) + I_o(t) \tag{9-22}$$

where

$$I_o(t) \equiv K_{FM}m_I(t)\left(\frac{1}{2\pi}\frac{d\Delta}{dt}\right)\left(\frac{1 + R_S\cos\Delta}{1 + R_S^2 + 2R_S\cos\Delta}\right) \tag{9-23}$$

We now wish to express the interference amplitude in a more usable form. To do this, we observe that the bracketed quantity in equation (9-23) is periodic in $\Delta[\equiv\Delta\omega t - \phi_S(t)]$, and proceed to express it as a Fourier series.

The basic details of this derivation are given in APPENDIX I, equations (I-188) and (I-208). The results are that the interference signal is found to consist of two types of normalized terms (i.e., **normalizing** with respect to K_{FM}) and is found to be

$$I(t) = I_1(t) + I_2(t) \tag{9-24}$$

where

$$I_{T1} \equiv \frac{1}{2} m_I(t)[1 + u(R_S)][\Delta f - S_K(t)] \tag{9-25}$$

and

$$I_{T2} \equiv \frac{m_I(t)}{2\pi}\left(\frac{d}{dt}\sum_{n=1}^{\infty}\frac{u(R_S)(-R_S)^{nu(R_S)}}{n}\sin[n\Delta\omega t - n\phi_S(t)]\right) \tag{9-26}$$

where

$$u(R_S) \equiv \begin{cases} +1 \text{ when } R_S < 1 \\ \;\;0 \text{ when } R_S = 1 \\ -1 \text{ when } R_S > 1 \end{cases} \tag{9-27}$$

Some comments are now in order concerning the above results. The inherently stronger of the two interference signals is $I_1(t)$, although when $R_S > 1$, it disappears altogether. (Also, when $R_S = 1$, its value is half what it is for $R < 1$.) Assuming that $R_S \leq 1$, $I_1(t)$ is seen to be the product of the interference binary waveform $m_I(t)$ with an audio frequency signal, which contains a dc term, Δf, and an ac term, $S_K(t)$, i.e., the signal modulation. Hence, the spectral power contained in this discriminator output signal resides for the most part within the low-pass filter bandwidth. That is, one can expect that most of the power in $I_1(t)$ will be passed to the receiver output. $I_2(t)$, on the other hand, can be characterized as a series of carriers located at Δf and its harmonics, where each such carrier is frequency modulated by some multiple of the desired modulation. Thus, let us consider the general nth term of the series of equation (9-26), i.e.,

$$I_n(t) = \left[\frac{1}{n}u(R_S)(-R_S)^{nu(R_S)}\right]\sin\left[n\Delta\omega t - n\phi_S(t)\right] \qquad (9\text{-}28)$$

From past observations, we can deduce three basic facts concerning this term:

1. It is a frequency modulated signal of carrier $n\Delta f$ and high modulation index.
2. It has a bandwidth 2nB, i.e., n times the transmission bandwidth of the system.
3. Its amplitude decreases progressively with increasing n for all R_S, except $R_S = 1$, at which value $A_N = 0$.

Hence, as the multiple order of the carrier frequency increases, the total signal power becomes lower and is spread over an increasingly wider bandwidth. For values of Δf on the order of B or less, this will result in considerable overlapping of the spectra of the individual terms of the series. This situation is depicted in Figure 9-3.

A meaningful index of the effects of pulse interference on the FM receiver is the total output power due to the individual signals $I_1(t)$ and $I_2(t)$.

Let us consider again the general nth term of the series of (9-26), i.e., $i_n(t)$ as given by equation (9-27). We will attempt

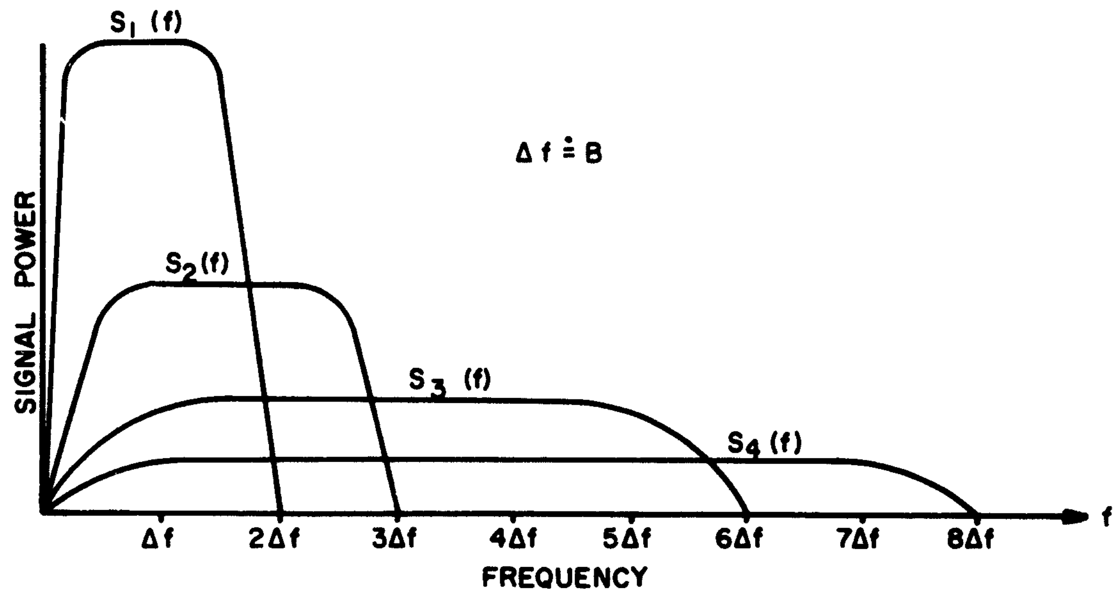


Figure 9-3. FM Output Spectrum

to synthesize a spectral power distribution for this signal based on a few fundamental facts. We begin by noting that, since we are concerned here only with the power spectrum, there is reciprocity between the cases $R_S > 1$ and $R_S < 1$; that is, for a particular $R = R_1$, the squared amplitude of $i_n(t)$ is the same as for $R = 1/R_1$. Hence, let us assume with no loss in generality that $R_S < 1$, so that the amplitude of $i_n(t)$ is $(R_S{}^n/n)$. Thus, its total power is given by

$$P_n = \frac{R_S{}^{2n}}{n^2} \overline{\sin^2[n\Delta\omega t - n\phi_S(t)]} = \frac{R_S{}^{2n}}{2n^2} \tag{9-29}$$

Furthermore, we know that the bandwidth of this signal is

$$(BW)_n = 2nB \tag{9-30}$$

where 2B is the system transmission bandwidth. Since we consider $S_K(t)$ to be an arbitrary modulating waveform, it is reasonable to

assume that the spectral power of $i_n(t)$ is uniformly distributed throughout the range within $\pm$ nB cps of $n\Delta f$. A convenient and realistic mathematical expression for this power spectrum would therefore be*

$$S_n(f) = A_n \frac{1}{1 + \left(\frac{f}{nB} - \frac{\Delta f}{B}\right)^2} \tag{9-31}$$

That is, $S_n(f)$ can be approximated by a fairly uniform and continuous distribution wherein most of its energy is contained within the region

$$n\Delta f - nB \leq f \leq n\Delta f + nB \tag{9-32}$$

In order for (9-30) to qualify as a legitimate spectrum approximation, A_n must be chosen so as to reflect the true total power of the signal. That is, if we require that

$$\int_{-\infty}^{\infty} S_n(f)\, df = P_n = \frac{R_S^{\,2n}}{2n^2}$$

then we obtain, using equation (9-30)

$$A_n = \frac{1}{2\pi B}\left(\frac{R_S^{\,2n}}{n^3}\right) \tag{9-33}$$

To complete this development, we make the assumption that the overlapping spectral components of $S_1(f)$, $S_2(f)$, etc., bear no phase correlation, so that their powers add directly. This is a quite valid assumption for any typical modulating signal, $S_K(t)$. Hence, we can give the total power spectrum for the series in equation (9-26) by

---

*We have here expressed the spectral distribution about $n\Delta f$ as a single-sided spectrum, on the assumption that the overlapping components of the double-sided spectrum are uncorrelated in phase. This assumption is quite reasonable for any practical modulating signal $S_K(t)$.

$$S(f) \equiv \sum_{n=1}^{\infty} S_n(f) = \frac{1}{2\pi B} \sum_{n=1}^{\infty} \frac{R_S{}^{2n}/n^3}{1 + \left[\frac{f}{nB} - \frac{\Delta f}{B}\right]^2} \tag{9-34}$$

We will now use the developments of the last section to determine the receiver output interference power due to $I_1(t)$ and $I_2(t)$, equations (9-25) and (9-26). Let us assume initially that $R_S < 1$, and consider the output power due to $I_1(t)$. Using equation (9-28), we can write $I_1(t)$ as

$$I_1(t) = \delta\Delta\omega + \sqrt{\delta(1-\delta)}\Delta\omega a(t) - m_I(t)S_K(t) \tag{9-35}$$

The first term is a dc component which is assumed to be removed from the receiver output by capacitive blocking. Hence, we will concern ourselves only with the latter two terms.

We could use equation (9-9) for the transfer function of the low-pass filter, but we will approximate H(jf) in this case by

$$H(jf) = \frac{1}{1 + (jf/f_d)} \tag{9-36}$$

using the assumptions that (1) β is substantially greater than f_d, and (2) the largest components of the power spectrum of the interference do not reside at frequencies above β. Both assumptions are totally reasonable for the case being considered. Combining equations (9-35) with (9-36), the ac interference power at the receiver output due to I is found to be

$$P_{I1} = \delta(1-\delta)\Delta\omega^2 \int_{-\infty}^{\infty} \frac{a(f)\ df}{1 + (f/f_d)^2} + \delta_{eff} \int_{-\infty}^{\infty} \frac{w(f)\ df}{1 + (f/f_d)^2} \tag{9-37}$$

where δ_{eff} is the effective duty cycle defined in APPENDIX I by equation (I-49) and w(f) is the power spectrum of $S_K(t)$. But, the second integral is the intended mean-square value of the frequency deviation which modulates the transmitted carriers. That is, if the mean-square angular frequency deviation is ω_{rms}^2, then ideally the system is designed so that the output signal power will be

$$P_S = \int_{-\infty}^{\infty} \frac{w(f)\,df}{1 + (f/f_d)^2} = \omega_{rms}^2 \tag{9-38}$$

Furthermore, if the peak angular frequency deviation is $2\pi B$, then we can approximate ω_{rms} by

$$\omega_{rms} = \frac{\sqrt{2}}{2}(2\pi B) \tag{9-39}$$

so that

$$P_S = \frac{1}{2}(2\pi B)^2 \tag{9-40}$$

The first integral of equation (9-37) is not as readily approximated. We have stipulated that the average pulse width of a(t) should not exceed $1/\beta_{IF}$ where β_{IF} is on the order of B. Hence, since $B \gg f_d$, the power spectrum $\alpha(f)$ will not be contained by the low-pass filter for all cases considered. We will resort here to the special case where $m_I(t)$ is a random telegraph signal, i.e., where $\delta = 0.5$ and $\alpha(f)$ is given by equation (I-41), and obtain the interference power P_{I1} for this case. We therefore obtain

$$P_{I1} = \frac{\sqrt{2}}{2}(\pi B)^2 \left[1 + \sqrt{2}(\Delta f/B)^2 \cdot \frac{\pi f_d \tau}{1 + \pi f_d \tau}\right] \tag{9-41}$$

Comparing this to the desired signal output power, equation (9-40), we obtain

$$\frac{P_{I1}}{P_S} = \delta_{eff}\left[1 + \sqrt{2}(\Delta f/B)^2 \cdot \frac{\pi f_d \tau}{1 + \pi f_d \tau}\right] \tag{9-42a}$$

$$= \frac{\sqrt{2}}{4}\left[1 + \sqrt{2}(\Delta f/B)^2 \cdot \frac{\pi f_d \tau}{1 + \pi f_d \tau}\right] \tag{9-42b}$$

Curves of (P_{I1}/P_S) vs. $(\Delta f/B)$, with $(\pi f_d \tau)$ as a parameter, are given in Figure 9-4. It should be remembered that this result applies only when $R_S < 1$. When $R_S > 1$, the $I_1(t)$ term is zero and

$P_{I1} = 0$. For the special case $R_S = 1$, the amplitude of I_{t1} reduces to one-half its value for $R_S < 1$, so that (P_{I1}/P_S) as given by equation (9-42) is reduced by a factor of 4.

To find the total interference power output due to $I_2(t)$, we will use equation (9-9) for the low-pass filter transfer function. From (I-210), the power spectrum can be shown to be

$$S_o(f) = (2\pi f)^2 \sum_{n=1}^{\infty} S_n(f) \qquad (9\text{-}43)$$

where the $(2\pi f)^2$ term is due to the differentiation of the series, and the individual $S_n(f)$ can be obtained from equation (9-34). (We are tacitly assuming here that $R_S < 1$, with the understanding that R_S is replaced by R_I in equation (9-34) when $R_S > 1$, and $S_n(f) = 0$ for the special case $R_S = 1$.) We now invoke the concept of effective duty cycle, and find the output power due to $I_2(t)$ to be

$$P_{I2} = \delta_{eff}(2\pi)^2 \int_{-\infty}^{\infty} |H^2(jf)| \sum_{n=1}^{\infty} S_n(f)\, df \qquad (9\text{-}44)$$

Using equations (9-34) and (9-39), and the fact that $\delta_{eff} = \sqrt{2}/4$ when $\delta = 0.5$, we find after making suitable manipulations that

$$\frac{P_{I2}}{P_S} = \frac{\sqrt{2}}{4\pi} \sum_{n=1}^{\infty} \frac{R_I^{2n}}{n^3} \int_{-\infty}^{\infty} \frac{x^2\, dx}{\left[1 + \left(\frac{B}{f_d}\right)^2 x^2\right]\left[1 + \left(\frac{B}{\beta}\right)^2 x^2\right]\left[1 + \left(\frac{x}{n} - \frac{\Delta f}{B}\right)^2\right]}$$

where $x = f/B$. (9-45)

If we were to obtain this quantity as a function, R, and $(\Delta f/B)$, we would have to assign values to (B/f_d) and (B/β). In order to estimate the magnitudes involved, let us consider a typical wide-band FM system, wherein $f_d \doteq 2100$ cps, $\beta \doteq 15{,}000$ cps, and $B \doteq 75{,}000$ cps; thus, we would have $(B/f_d) \doteq 35.7$, and $(B/\beta) \doteq 5.0$. The maximum values of P_{I2} are in general obtained for $\Delta f/f_d \ll 1$ and $R_I \doteq 1$. Hence, for the assigned values of f_d, β, and B, the

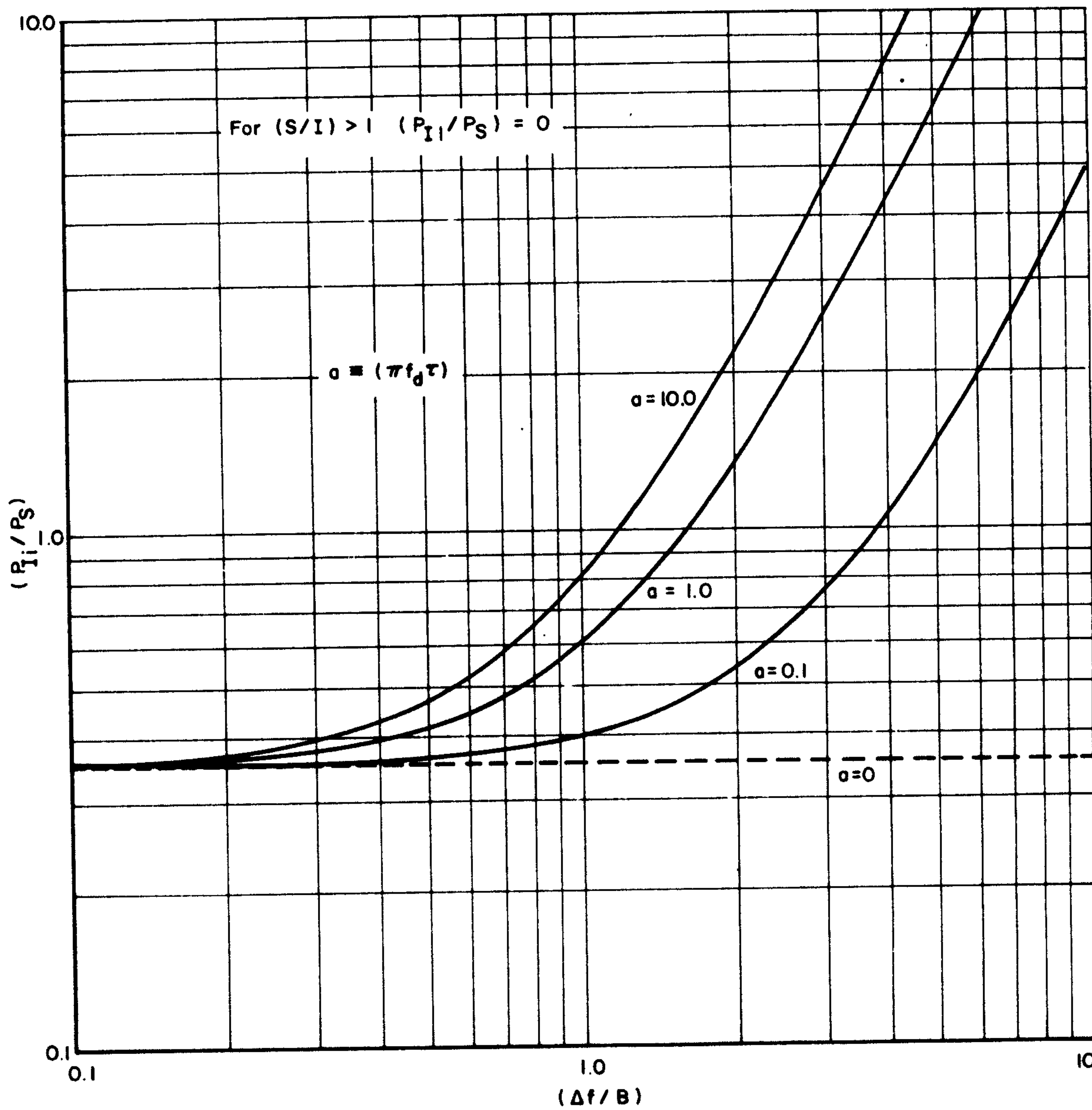


Figure 9-4. Interference Output Power P_{I1} Due to Pulse Interference Input ($\delta = 0.5$, S/I < 1)

maximum value of equation (9-45) obtained would be on the order of

$$\left(\frac{P_{I2}}{P_S}\right)_{max} \doteq 2 \times 10^{-5} \qquad (9\text{-}46)$$

It is thus seen that for this situation the output interference power due to $I_2(t)$ is negligible, in a typical wideband FM system, compared with the output signal power.

Let us now consider an FM system which uses no pre-emphasis and de-emphasis, and for which (B/β) can be any value greater than unity. In this case, the receiver low-pass filter can be assumed to be a simple RC circuit with a cutoff frequency of β. We can therefore write the integral of equation (9-45) as

$$I_n = \int_{-\infty}^{\infty} \left(\frac{\beta}{B}\right)^2 \left(\frac{(Bx/\beta)^2}{1 + (Bx/\beta)^2}\right) \cdot \frac{dx}{1 + \left(\frac{x}{n} - \frac{\Delta f}{B}\right)^2} \qquad (9\text{-}47)$$

For $(B/\beta) \geq 1$, the bracketed term inside the integral is close to unity over most of the significant range of integration. Hence we can approximate

$$I_n \doteq \left(\frac{\beta}{B}\right)^2 \int_{-\infty}^{\infty} \frac{dx}{1 + \left(\frac{x}{n} - \frac{\Delta f}{B}\right)^2} = n\pi\left(\frac{\beta}{B}\right)^2 \qquad (9\text{-}48)$$

The consequence of this approximation is that, for the extreme case where $(B/\beta) = 1$ and $(\Delta f/B) \ll 1$, the error involved in calculating P_{I2} is high, i.e., conservative by less than 3 db. Applying this result to equation (9-45) we obtain

$$\frac{P_{I2}}{P_S} = \frac{\sqrt{2}}{4\pi}\left(\frac{\beta}{B}\right)^2 \sum_{n=1}^{\infty} \frac{R_I^{2n}}{n^2}, \qquad (R<1) \qquad (9\text{-}49)$$

For $R_I > 1$, we merely replace R_I in equation (9-49) by R_S. Curves of (P_{I2}/P_S) vs. R_I, with (B/β) as a parameter, are given in Figure 9-5.

DISCUSSION OF RESULTS

The interference output of the FM receiver consists essentially of two signals. One of these, $I_1(t)$, is directly related to the desired modulation, $S_K(t)$, and has a total power represented by the

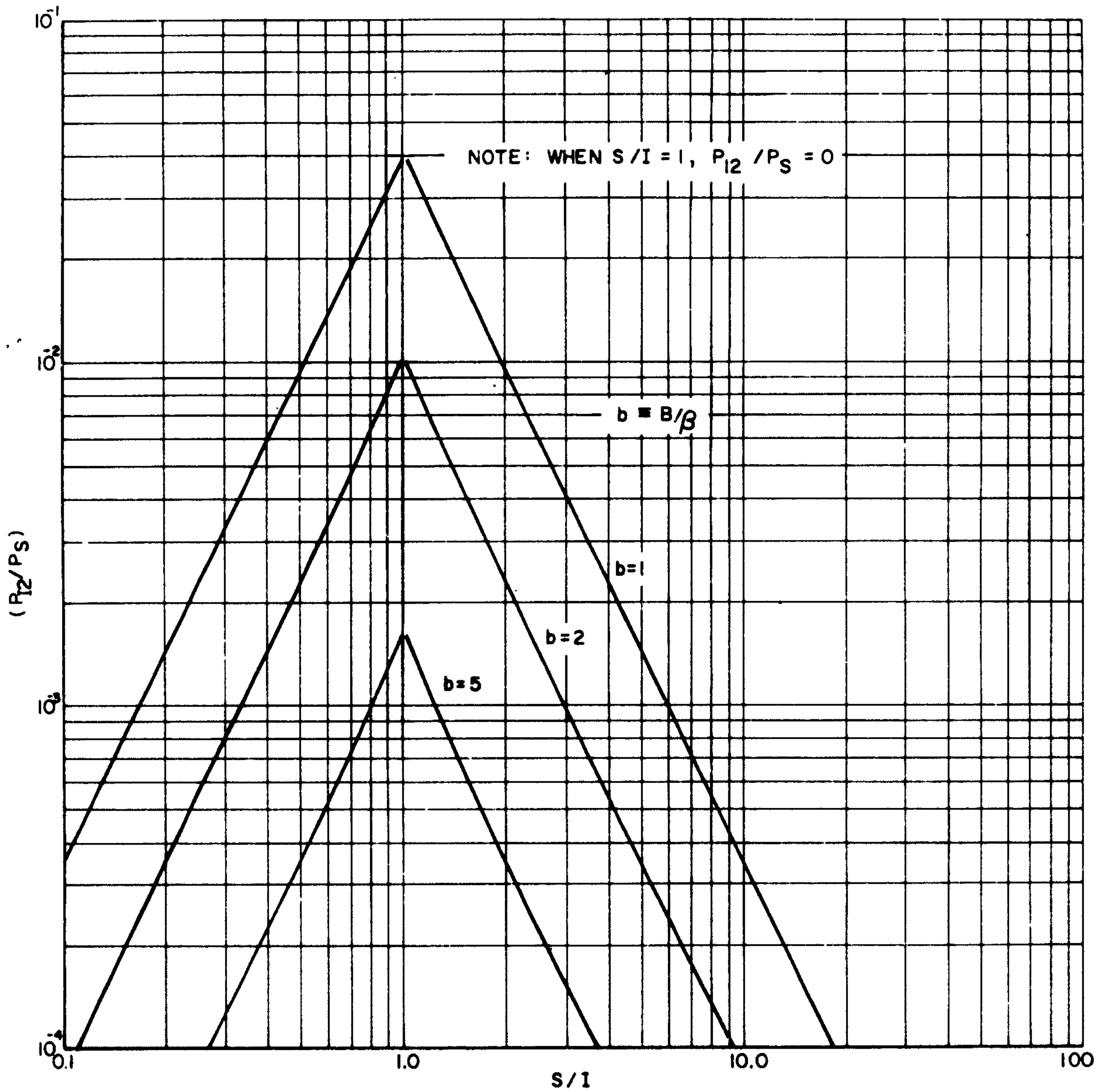


Figure 9-5. Interference Output Power, P_{I2}, Due to Pulse Interference Input (δ = 0.5, No De-emphasis Filtering)

curves of Figure 9-4; the other, $I_2(t)$, is a series of frequency-modulated carriers at Δf, $2\Delta f$, ..., $n\Delta f$, etc., and has a total power represented by the curves of Figure 9-5. The major analytical restrictions employed in achieving these results have been the assumptions that:

1. The system is wideband, i.e., the maximum frequency deviation exceeds the modulation bandwidth.
2. The input signal is detected by means of ideal limiting and frequency discrimination.
3. The pulse interference has a duty cycle of 0.5, and a sideband distribution akin to that of a random telegraph signal.

The first assumption has been made more in the interest of realism than mathematical simplicity, and requires no further justification. The second assumption may break down for values of $\Delta f/B$ greater than the order of 2. The last assumption is very convenient in that it permits us to employ the concept of effective duty cycle with at most a 1.5-db error in estimating output interference power. This assumption is justified by the facts that (1) such a signal is, in its essential features, typical of the pulse interference to which an FM system might be subjected, and (2) a duty cycle of 0.5 leads to a worst-case estimate of the effects of pulse interference (as compared with $\delta < 0.5$).

The results of the above analysis indicate that $I_1(t)$ disappears when $(S/I) > 1$, reaffirming the notion of the threshold effect in FM receivers, and that the output power due to $I_2(t)$ is negligible in typical FM systems using pre-emphasis and de-emphasis. The curves of Figure 9-5 are presented for the case where no pre-emphasis-de-emphasis is used.

The basic problem remaining for degradation analysis is to then obtain performance curves for undesired signals of approximately a 50% duty cycle.

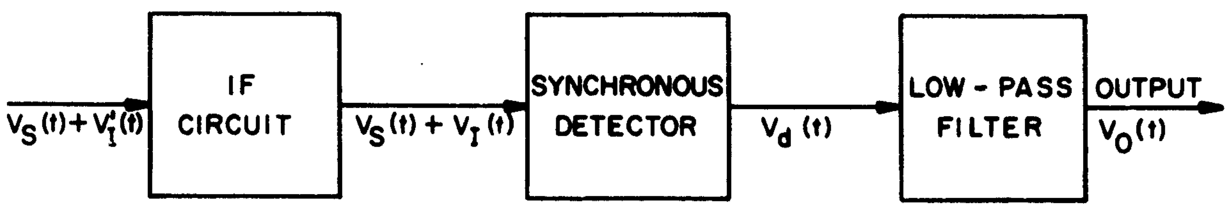
The results of this section generally apply to both analog and voice modulation.

SECTION 10

A DESIRED SSB SIGNAL INTERFERED WITH BY AN UNDESIRED PULSED SIGNAL

INTRODUCTION

The portion of the single sideband receiver to be treated in the foregoing analysis is depicted in Figure 10-1.



$v_S(t)$ = Desired signal

$v_I'(t)$ = IF input undesired signal

$v_I(t)$ = IF output undesired signal

$v_d(t)$ = Detector output signal

$v_o(t)$ = Low-pass filter output signal.

Figure 10-1. SSB Receiver, Portion Analyzed

The assumptions used for the following receiver subsections and the input signals can be stated as follows:

1. The input desired signal is given by

$$v_S(t) = A_S \sum_{k=1}^{N} m_{SK} \cos (\omega_o + \omega_{SK})t \qquad (10\text{-}1)$$

where the desired modulation is

$$m_S(t) = \sum_{k=1}^{N} m_{SK} \cos(\omega_{SK} t) \quad , \qquad \text{all } m_{SK} \leq 1 \quad (10\text{-}2)$$

2. System noise is assumed small in comparison with the desired signal and is consequently neglected.

3. The input interference signal is given by

$$v_I'(t) = A_I' m_I'(t) \cos[(\omega_o + \Delta\omega)t + \theta_I)] \qquad (10\text{-}3)$$

where $m_I'(t)$ is a binary on-off waveform of unit amplitude. It is assumed that the average width of the pulses of $m_I'(t)$ obeys the inequality

$$\beta_{IF}\tau \geq 1 \qquad (10\text{-}4)$$

where β_{IF} is the IF half-bandwidth. No restrictions are placed upon the values of A_I' or $\Delta\omega$ within the assumptions of APPENDIX III.

4. The IF amplifier is assumed to pass $v_S(t)$ without modification and to alter $v_I'(t)$ by an amplitude factor $g(\Delta f)$ which depends on the IF selectivity curve, where $g(0) = 1$. Hence, the interference signal at the IF output is

$$v_I(t) = g(\Delta f)\, v_I'(t) = A_I m_I(t) \cos[(\omega_o + \Delta\omega)t + \theta] \qquad (10\text{-}5)$$

where

$$A_I = A_I' g(\Delta f) \qquad (10\text{-}6)$$

The analysis will treat A_I as the effective amplitude of the input interference. The transformation from $m_I'(t)$ to $m_I(t)$ is discussed in APPENDIX III. Since this transformation is a function of the particular IF transfer function and Δf that is chosen, it will be

sufficient for this discussion to leave the transformation symbolically as $m_I(t)$. For the particular case of a symetrical transfer function, they are of the same form.

Also, the duty cycle of the pulse interference is assumed to be

$$\delta \; [\equiv \overline{m_I(t)}] \approx 0.5 \qquad (10\text{-}7)$$

5. The synchronous detector is assumed to be an ideal frequency translator, as discussed in APPENDIX I, equation (I-220), heterodyning the input spectrum downwards by an amount f_o, with no change in amplitude.

6. The low-pass filter is a conventional RC circuit whose transfer function is given by

$$H(jf) = \frac{1}{1 + j(f/\beta)} \qquad (10\text{-}8)$$

where β is the audio bandwidth of the modulating waveform $m_S(t)$.

DERIVATION OF THE RECEIVER OUTPUT SIGNAL

Since the detector is assumed to be an ideal synchronous detector, the corresponding derivation used in (I-224b) and (I-231) of APPENDIX I applies. Using the expressions given for the input signal waveforms, equations (10-1) and (10-6), the normalized detector output is thus found to be

$$V_d(t) = \frac{A_S}{2} \sum_{k=1}^{N} m_{SK} \cos \omega_{SK} t + \frac{A_I}{2} m_I(t) \cos(\Delta\omega t + \theta_I)$$

$$= \frac{A_S}{2} m_S(t) + \frac{A_I}{2} m_I(t) \cos(\Delta\omega t + \theta_I) \qquad (10\text{-}9)$$

Thus, the detector output consists of the desired demodulated signal and the pulse interference signal centered at Δf.

Processing this signal through the low-pass filter, we find the receiver output signal to be

$$V_o(t) = S_o(t) + I_o(t) \tag{10-10}$$

$$S_o(t) \equiv \pi\beta \int_{-\infty}^{t} A_S m_S(\tau) \exp\left\{-2\pi\beta(t-\tau)\right\} d\tau \tag{10-11}$$

$$I_o(t) \equiv \pi\beta \int_{-\infty}^{t} A_I m_I(\tau) \cos(\Delta\omega\tau + \theta) \exp\left\{-2\pi\beta(t-\tau)\right\} d\tau \tag{10-12}$$

and $2\pi\beta \exp(-2\beta t)$ is the impulse response of the filter. Whereas equation (10-11) and (10-12) are the formal solutions for the desired and interference signals at the receiver output, they do not provide a useful representation for purposes of analysis. A possible method for characterizing the output interference in a more meaningful way will be demonstrated in the next section.

The unique feature of a SSB receiver using synchronous detection is that the output interference signal is independent of the strength and form of the desired signal. In the case considered here, the relative output interference signal is identical to the input interference signal, except for the translation of the carrier frequency to Δf, and possible waveform modification due to the low-pass filter. Since we are treating only the cases where $\beta t \geq 1$, this modification will consist primarily of amplitude reduction rather than waveshape distortion. Hence, the most significant characteristics of the interference output would seem to be merely its carrier frequency Δf and total power, P_I.

In order to find P_I, we invoke the assumption $\beta t \geq 1$, which enables us to approximate the low-pass filter interference output by

$$I_o(t) \doteq \frac{1}{2} \frac{A_I m_I(t)}{\sqrt{1 + (\Delta f/\beta)^2}} \cos(\Delta\omega t + \theta + \phi) \tag{10-13}$$

where ϕ is the phase shift through the filter at frequency Δf. The total power contained in this signal is

$$P_I = \frac{A_I^2/4}{1 + (\Delta f/\beta)^2} \left\{\overline{m_I^2(t) \cos^2 (\Delta\omega t + \theta)}\right\}$$

$$= \frac{1}{8} \frac{\delta A_I^2}{1 + (\Delta f/\beta)^2} \qquad (10\text{-}14)$$

This result presupposes that the pulses of $m_I(t)$ do not occur at a periodic rate which is an integral submultiple of Δf.

If P_I were derived rigorously, i.e., without the approximation implied by equation (10-13), the result obtained would differ from equation (10-14) by at most -0.45 db (for $\delta \approx 0.5$), this discrepancy occurring in the extreme case where $\Delta f/\beta \ll 1$ and the ac term of $m_I(t)$ has its spectrum uniformly distributed in the range $0 \leq f \leq \beta$.

The desired signal component of the detector output can be assumed to be passed with virtually no modification by the low-pass filter. Hence, from equation (10-9), we can approximate the desired signal power at the receiver output by

$$P_S \doteq \frac{A_S^2}{4} \overline{m_S^2(t)} \qquad (10\text{-}15)$$

The output signal-to-interference power ratio, from equation (10-14) and (10-15), is therefore given by

$$\left(\frac{P_I}{P_S}\right)_{out} = \left(\frac{1}{2} \cdot \frac{\delta}{\overline{m_S^2(t)}}\right) \cdot \frac{R_I^2}{1 + (\Delta f/\beta)^2} \qquad (10\text{-}16)$$

But, from equations (10-1) and (10-6) we can see that the input powers due to the desired and interfering signals are

$$A_S^2\overline{m_S^2(t)} \quad \text{and} \quad \delta A_I^2/2$$

respectively. Inserting these results into equation (10-16) we obtain

$$R = \frac{(P_S/P_I)_{in}}{(P_S/P_I)_{out}} = \frac{1}{1 + (\Delta f/\beta)^2} \qquad (10\text{-}17)$$

This simple relationship is shown graphically in Figure 10-2.

DISCUSSION OF RESULTS

The most important restrictions placed upon the pulse interference analysis for the SSB receiver have been the assumptions that:

1. $\beta t > 1$.
2. The synchronous detector is ideal.

The first condition is not at all crucial to the tractability of the analysis, and could be violated with only a slight increase in mathematical complexity. It merely permits us to treat the effects of the low-pass filter (and tacitly, of the IF amplifier as well) in a mathematically simple way. The second assumption is reasonable under normal operating conditions.

The basic problem remaining for degradation analysis is to then obtain performance curves for undesired signals of approximately a 50% duty cycle

The results of this section generally apply to both analog and voice modulation.

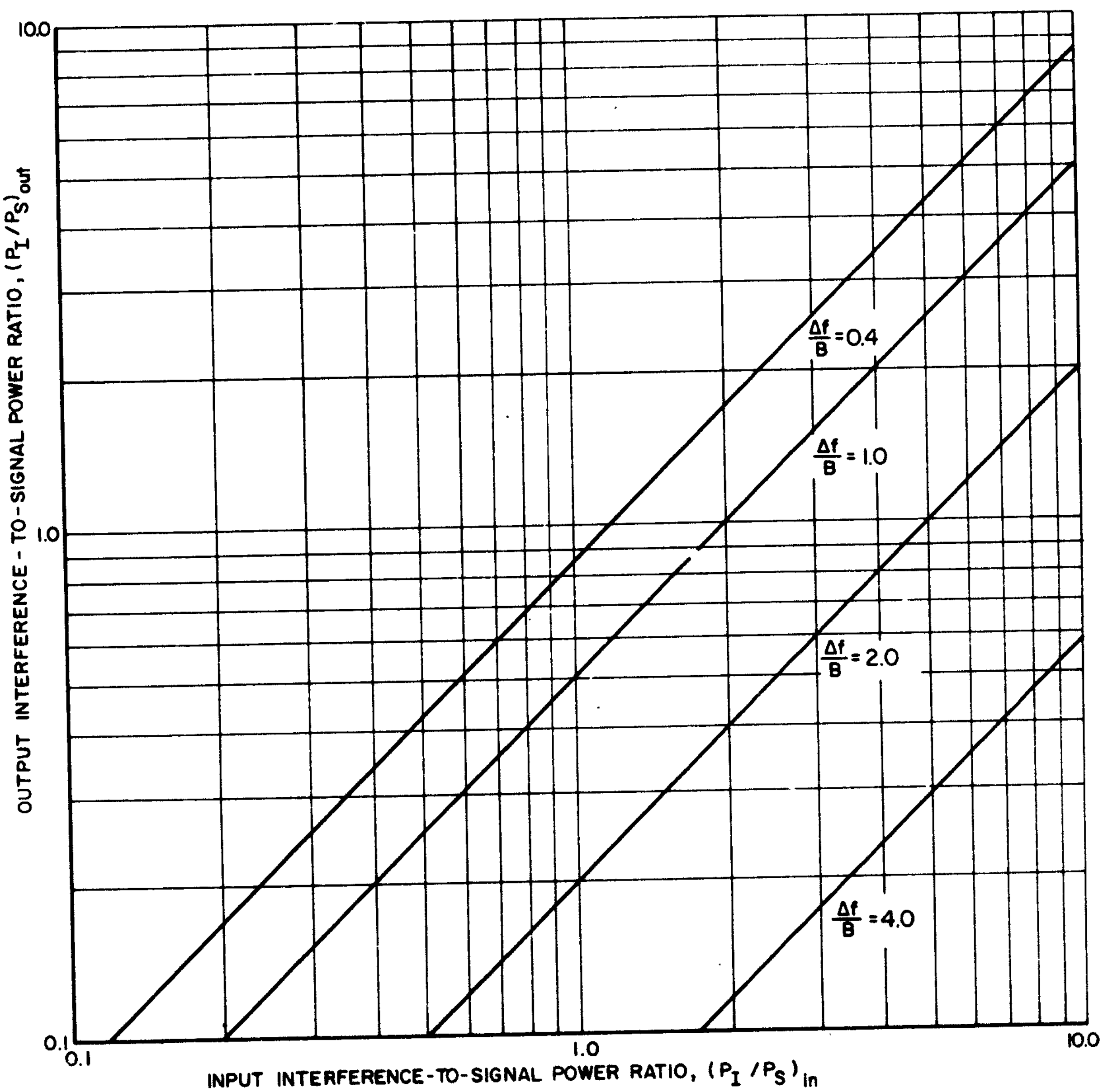


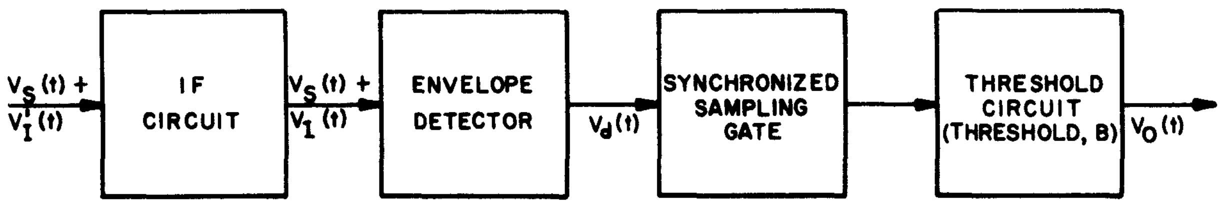
Figure 10-2. Output Interference Power Due to Pulse Interference ($\beta\tau \geq 1$)

SECTION 11

A DESIRED PULSED SIGNAL INTERFERED WITH BY AN UNDESIRED PULSED SIGNAL

INTRODUCTION

The portion of the digital pulsed carrier receiver to be treated in the following analysis is depicted in Figure 11-1.



$v_S(t)$ = Desired signal

$v_I'(t)$ = IF input undesired signal

$v_I(t)$ = IF output undesired signal

$v_d(t)$ = Detector output signal

$v_o(t)$ = Low-pass filter output signal

Figure 11-1. Digital Pulsed Carrier, Portion Analyzed

The underlying assumptions to be employed with respect to the individual receiver subsections and the input signals can be stated as follows:

1. The input desired signal is given by

$$v_S(t) = A_S m_S(t) \cos \omega_o t \tag{11-1}$$

where $m_S(t)$ is a random binary waveform which is either 0 or 1 at all times but which changes states only at integral multiples of T_B, i.e., at $t = T_B$ and/or $t = 2T_B$ and/or $t = 3T_B$, etc. That is, we assume a synchronous digital system of bit-width, T_B, where the occurrence of 1's or 0's in each bit interval is completely random.

2. The input interference signal is given by

$$v_I'(t) = I' m_I'(t) \cos[(\omega_o + \Delta\omega)t + \theta] \qquad (11\text{-}2)$$

where $m_I'(t)$ is a binary waveform of unit amplitude, average pulse width, t, and duty cycle, δ. We will consider here all possible values of δ and two possible cases for t, namely $t = T_B$, and $t >> T_B$. These are the normal situations encountered in narrow band communication systems. The radar problem of a narrow band pulse impinging on a wideband system (i.e., the familiar off-tuned pulse effect) will not be considered.

3. In addition to desired signal plus interference, the input is assumed to consist of white, gaussian receiver noise. The resultant narrow-band gaussian noise voltage at the IF output is assumed to have a mean-square value, N.

4. The IF circuit is assumed to have a steady-state voltage gain, $g(\Delta f)$, where $g(0) = 1$, and where the 3-db half-bandwidth, β_{IF}, of this gain characteristic obeys the relationship

$$\beta_{IF} T_B = 0.6 \qquad (11\text{-}3)$$

This is a typical design value for conventional IF circuits because it yields the maximum peak signal-to-rms-noise ratio for desired pulses. From an analytical standpoint it permits us to approximate the IF pulse response in the manner depicted in Figure 11-2. Thus, the IF output builds up linearly in time, T_B, to its steady-state value, $A' g(\Delta f)$, and decays linearly at the same rate to zero, commencing with the trailing edge of the applied pulse. We will define

$$A_I \equiv A_I' \, g(\Delta f) \qquad (11\text{-}4)$$

and treat A_I as the effective amplitude of the input interference.

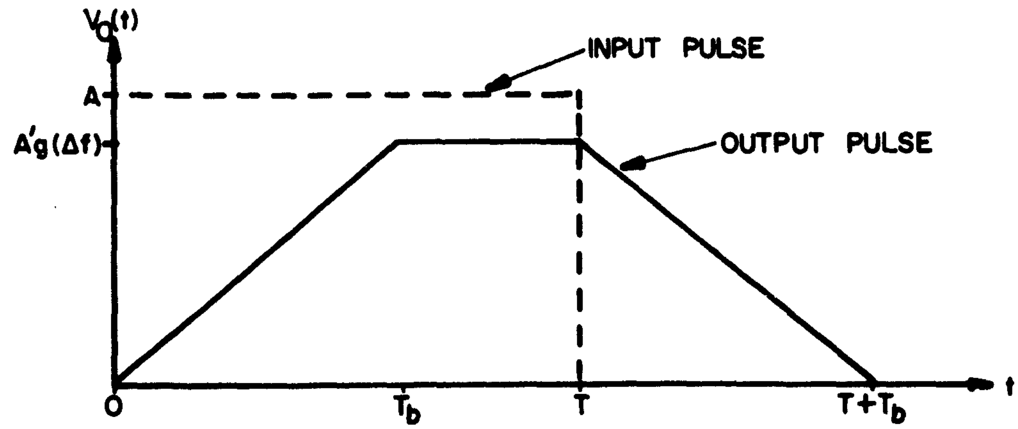


Figure 11-2. Approximation of IF Pulse

5. The digital pulsed carrier system is assumed to be bit synchronous, and the receiver is assumed to be in synchronization with the desired signal. Thus, at the end of each bit interval, the envelope of the IF output is sampled and compared to a fixed threshold voltage, V_B. The receiver output for that interval is derived on the basis of the following rule: If the sampled envelope exceeds V_B, the receiver reacts as though a 1 was transmitted and correspondingly generates a 1 at the output; if the sampled envelope does not exceed V_B, the receiver reacts as though a 0 was transmitted and correspondingly generates a 0 at the output. Under ideal conditions, therefore, the binary waveform which modulates the carrier is reproduced at the receiver output, that is

$$v_o(t) = m_S(t) \quad \text{(Ideal Conditions)} \qquad (11\text{-}5)$$

6. The detection scheme described above can be implemented by means of the envelope detector, sampling gate, and threshold circuit indicated in Figure 11-1. The envelope sampling time at the end of each bit interval is assumed to be quite small compared to the width, T_B, of the interval, and all circuits are assumed to perform ideally.

INDICES OF SYSTEM PERFORMANCE

Bit Error Probabilities. The pulsed carrier receiver is subject to two kinds of bit errors, namely, the error of omission, which occurs when a desired signal pulse fails to be detected, and the error of commission, which occurs when a signal pulse is erroneously declared. The probabilities for these errors can be denoted by the conditional probabilities $P(0|1)$ and $P(1|0)$, respectively, where $P(0|1)$ is the probability of a 0 being declared when in fact a 1 is present, and conversely for $P(1|0)$.

The two error probabilities can be related to the probability of pulse detection, P_D, by the simple expressions

$$P(0|1) = 1 - P_D(A_S, A_I, N) \tag{11-6}$$

and

$$P(1|0) = P_D(S = 0, A_I, N) \tag{11-7}$$

where the dependence of P_D on A_S, A_I and N is indicated by the notation used.

It must be considered that for any bit interval in which a desired signal pulse is present, the conditional probability $P(0|1)$ will depend upon the interference state during that interval, that is, upon the amplitude, phase, and fractional time of occurrence during the interval; similarly for $P(1|0)$ when no desired signal pulse is present. But, such interference parameters as relative phase and fractional time of occurrence within the interval are statistical quantities which vary from bit interval to bit interval. Hence, $P(0|1)$ and $P(1|0)$ are statistical quantities which must be averaged over all possible interference states in order to obtain the expected error probabilities per bit. We will denote such an ensemble average for $P(0|1)$ by ϵ_1, and the corresponding average for $P(1|0)$ by ϵ_2, i.e.,

$$\epsilon_1 \equiv \overline{P(0|1)} = 1 - \overline{P_D(S,I,N)} \tag{11-8}$$

$$\epsilon_2 \equiv \overline{P(1|0)} = \overline{P_D(S=0,I,N)} \tag{11-9}$$

Qualitative Effects of Pulse Interference. If the victim receiver is part of a communication link receiving digital information or pulse coded analog information, the effect of pulse inter-

ference is to modify the expected probabilities for the two types of bit errors. In general, the interference will reduce the errors of omission and increase the errors of commission, where the increase in the latter is invariably more pronounced than the reduction in the former. Therefore, the net change in average error rate due to pulse interference will be positive for all practical systems.

The net amount of information loss due to pulse interference will depend greatly on the type of coding employed and also on the distribution of errors. For instance, if the errors are fairly uniformly distributed, the resultant message degradation will be proportionately smaller than the raw error rate, if an error correcting code has been used. If the errors are clustered in bursts, the message degradation may be quite serious even with relatively small average error rates, since whole words or phrases (or their message equivalents) may be obliterated rather than only individual characters or symbols. However, degradation due to such an error distribution may also be reduced by the use of burst-error correcting codes. Hence, no broad generalizations can be made about the effects of pulse interference on pulse communication systems without specific knowledge of the characteristics of the systems and the distribution of errors.

Bit Error Rate. As the discussion above has indicated, the question of how much degradation results in a system due to a given level of interference cannot be answered in general. However, some objective parameters can be found on a general basis which will serve as the input information for a detailed study of the degradation of a particular system. To this end, we will find the expected probability of error per bit for errors of omission and errors of commission. These probabilities of error can then be used to make an analysis of the overall message loss.

In some cases, particularly for systems using relatively unsophisticated codes, a useful index for evaluating information loss is the overall average error rate, which can be derived as follows: For any given bit interval, wherein there is no a priori knowledge of the signal state, the average error probability will be

$$\varepsilon_o = \overline{P(0|1)}\,P(1) + \overline{P(1|0)}\,P(0) \qquad (11\text{-}10)$$

where P(1) is the probability that the desired signal in that interval is a 1, and P(0) is the probability that the desired signal is a 0. But, P(1) is merely the average fraction of time occupied by the pulses of $m_S(t)$, i.e., the duty cycle, δ_S, of the desired

signal. Thus, using ε_1 and ε_2 for the expected error probabilities, we obtain

$$\varepsilon_o = \delta_S \varepsilon_1 + (1 - \delta_S)\varepsilon_2 \tag{11-11}$$

for the average error probability per bit. Multiplying this by the bit rate we obtain the average bit error rate,

$$r = \frac{1}{T_B} [\delta_S \varepsilon_1 + (1 - \delta_S)\varepsilon_2] \tag{11-12}$$

A possibly more general approach is to define an effective, or weighted bit error probability, i.e.,

$$\varepsilon_w \equiv w\varepsilon_1 + (1 - w)\varepsilon_2 \tag{11-13}$$

where ε_1 and ε_2 are weighted differently either because one has an opportunity to occur more frequently than the other, or is intrinsically more important in terms of information loss, or for some combination of these two reasons. The corresponding weighted error rate is

$$r_w = \frac{1}{T_B} [w\varepsilon_1 + (1 - w)\varepsilon_2] \tag{11-14}$$

In a typical digital system, wherein the 1's and 0's are equivalent with respect to frequency of occurrence and intrinsic importance, we would have $\delta = \omega = 1/2$ for which case equations (11-12) and (11-14) reduce to

$$r = r_w = \frac{1}{2T_B} (\varepsilon_1 + \varepsilon_2) \tag{11-15}$$

Analysis To Be Performed. From equations (11-8) and (11-9) we see that ε_1 and ε_2 can be derived strictly from the expected values of detection probability P_D, where ε_2 is obtained for the special case $A_S = 0$. Accordingly, the analysis will be performed in two basic steps, as follows:

1. The detection probability will be derived generally in terms of the pertinent parameters of the input signals and system. This will be done by characterizing the IF output during

the sampling interval as a fixed-amplitude sinusoid-plus-noise, and using available methods for determining the probability distribution of the sampled envelope and its corresponding probability of exceeding the threshold, V_B.

2. An approximation will then be employed for obtaining the ensemble average of the detection probability over all possible states of the interference signal.

The expected error probabilities will then be obtained from equations (11-8) and (11-9). These steps will first be performed for the case $\tau = T_B$, and the results so derived will then be applied to the case $\tau >> T_B$.

DERIVATION OF EXPECTED BIT ERROR PROBABILITIES

The so-called probability of detection, P_D, for a given bit interval is the probability that the sampled envelope will exceed the threshold voltage V_B. Let us denote the value of the sampled envelope by V and its probability density function by p(V), where

$$\int_0^\infty p(V)\, dV = 1$$

Then P_D is given simply by

$$P_D = \int_{V_B}^\infty p(V)\, dV \tag{11-16}$$

In order to find P_D, therefore, we must first find the probability density function for the sampled envelope. We will do this by developing the relationship between the receiver input components (signal, interference, and noise) and V.

To begin, let us consider the bit interval from t = 0 to $t = T_B$, and assume that a desired signal pulse of amplitude A_S appears at the IF input during this interval (where $A_S = 0$ corresponds to the no-signal case). Further, let us consider an input rectangular interference pulse of amplitude A_I, carrier frequency $(f_o + \Delta f)$, and width T_B, a portion ξT_B of which is contained within the specified bit interval. Thus, an input interference

pulse of width T_B which begins either at $t = -(1 - \xi)T_B$ or at $t = +(1 - \xi)T_B$ will occupy a portion ξT_B of the interval $0 \leq t \leq T_B$ and therefore satisfy the above condition. Since such a pulse occupies a fraction, ξ, of the specified interval, we will hereafter refer to ξ as the interference occupation parameter.

Because of the linearity of the IF circuit, we can invoke the principle of super-position and give its output in the general form

$$v(t) = n(t) + \{s(t) + i(t)\} \qquad 0 \leq t \leq T_B \tag{11-17}$$

where

n(t) is the narrowband gaussian voltage at the IF output, having mean-square value N;

s(t) is the IF output due to the desired signal pulse at the input; and

i(t) is the IF output due to the input interference signal.

The bracketed term of equation (11-17) is a deterministic signal obtained from the superposition of $v_S(t)$ and $v_I(t)$ at the IF input. The envelope of this signal will therefore fluctuate at a rate no greater than the order of Δf, i.e., the difference frequency between the desired and interfering carriers. We can reasonably assume, then, that the sampling time is short compared to the period of these envelope fluctuations, i.e., that $[s(t) + i(t)]$ is virtually a fixed-amplitude sinusoid during the length of the sampling period. In order to determine this amplitude, we shall invoke the IF response approximation depicted in Figure 11-2. Thus, we find s(t) to be

$$s(t) = A_S \frac{t}{T_B} \cos \omega_0 t \tag{11-18}$$

Near the end of the bit interval, i.e., within the sampling interval, which is short compared to T_B but long compared to the period of $\cos(\omega_0 t)$, s(t) is therefore given by

$$s(t) = A_S \cos \omega_0 t \qquad T_B \gg (T_B - t) \gg \frac{2\pi}{\omega_0} \tag{11-19}$$

In order to find i(t), we use the same approximation and find that, regardless of whether the interference starts at $t = -(1 - \xi)T_B$ or $t = +(1 - \xi)T_B$, i(t) can be given at the end of the bit interval by

$$i(t) = \xi A_I \cos [\omega_o t + (\Delta\omega t + \theta_I)] \tag{11-20}$$

$$T_B \gg (T_B - t) \gg \frac{2\pi}{\omega_o}$$

This result is demonstrated in Figure 11-3.

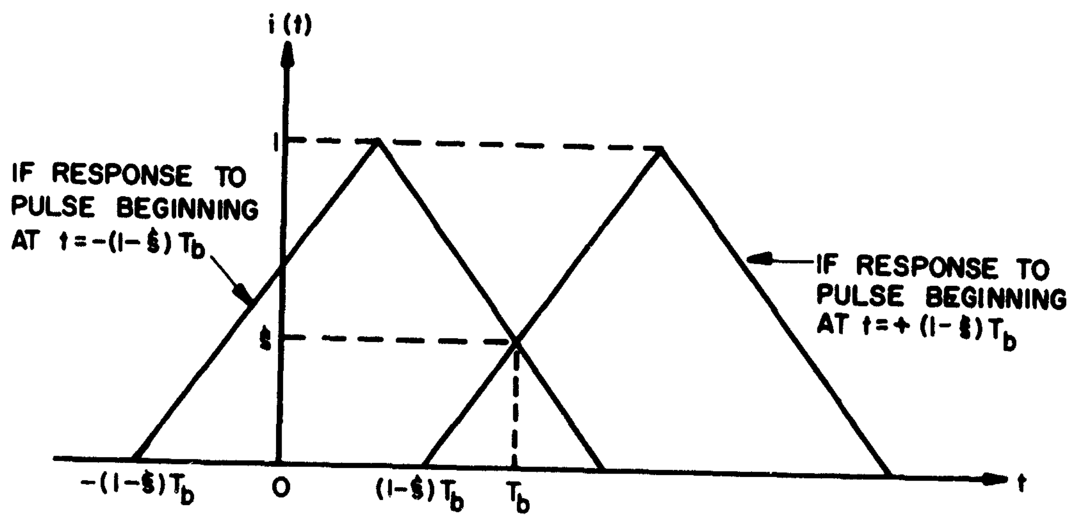


Figure 11-3. IF Response

Combining equations (11-19) and (11-20), and assuming that the sampling period is short compared to $1/\Delta f$, we can accurately obtain the sum of s(t) and i(t) during this period by use of the law of cosines. Thus,

$$s(t) + i(t) = E \cos (\omega_o t + \alpha) \tag{11-21}$$

$$\frac{2\pi}{\Delta\omega} \gg (T_B - t) \gg \frac{2\pi}{\omega_o}$$

where

$$E \equiv [A_S{}^2 + (\xi A_I)^2 + 2\xi A_I A_S \cos\phi]^{1/2} \tag{11-22}$$

$$\phi \equiv \Delta\omega t + \theta_I \tag{11-23}$$

and α is a complicated function of various signal parameters which is of no concern here.

Thus, we see that, within the duration of the sampling period, the deterministic component of the IF output can be given as a fixed-amplitude sinusoid of magnitude E, defined by equation (11-22). When no noise is added to this signal $[n(t) = 0]$, E is the value of the envelope observed by the sampling gate at $t = T_B$. When noise of mean-square value N is present, the sampled envelope becomes a statistical variable whose probability density function can be shown to be (reference 2):

$$p(V) = \frac{V}{N} \exp\left[-\frac{V^2 + E^2}{2N}\right] I_o\left[\frac{VE}{N}\right] \tag{11-24}$$

where $I_o(\ )$ is the modified Bessel function of zero order. It should be noted that, for a given bit interval, p(V) depends on the values of the statistical quantities ξ and $\cos\phi$ for that interval.

Now combining equation (11-24) with equation (11-16), we obtain the following expression for detection probability:

$$P_D = 2\int_{V_{Bn}}^{\infty} \rho \exp(-\rho^2 - E_n{}^2) I_o(2\rho E_N)\, d\rho \equiv Q(\sqrt{2}\, E_N, \sqrt{2}\, V_{Bn}) \tag{11-25}$$

where

$$E_N{}^2 \equiv S_N{}^2 + \xi^2 I_n{}^2 + 2\xi S_n I_n \cos\phi \tag{11-26}$$

$$S_n{}^2 \equiv S^2/2N \tag{11-27}$$

$$I_n{}^2 \equiv I^2/2N \tag{11-28}$$

and

$$V_{Bn} \equiv V_B/\sqrt{2N} \tag{11-29}$$

Thus, the detection probability is the so-called Q-function of $\sqrt{2}\, E_n$ and $\sqrt{2}\, V_{Bn}$, which has been tabulated by the RAND Corporation (3).

General expressions for the two types of errors can now be written in terms of the desired signal and interference signal amplitudes, the occupation parameter, the desired-to-interference signal relative phase angle, and the threshold level, using the above results with equation (11-6) and (11-7). The probability of an error of omission is given by

$$P(0|1) = 1 - 2\int_{V_{Bn}}^{\infty} \rho \exp\left[-(\rho^2 + S_n^2 + (\xi I_n)^2 + 2\xi I_n S_n \cos\phi\right] I_o \left[2\rho\sqrt{S_n^2 + (\xi I_n)^2 + 2\xi I_n S_n \cos\phi}\right] d\rho \qquad (11\text{-}30)$$

The probability of an error of commission is

$$P(1|0) = 2\int_{V_{Bn}}^{\infty} \rho \exp\left[-(\rho^2 + \xi^2 I_n^2)\right] I_o\,(2\rho\xi I_n)\, d\rho \qquad (11\text{-}31)$$

The quantities that we actually desire to compute are the ensemble averages of the two types of error over the two random variables ξ and ϕ in the expressions. Since ϕ occurs only as the argument of the cosine function, it will be convenient to consider $\mu \equiv \cos\phi$ the random variable. Moreover, since the random variables appear in the expressions for the error probabilities, equations (11-30) and (11-31), only in the probability of detection, P_D, the averaging need be carried out only for P_D, i.e.,

$$\epsilon_1 = \overline{P(0|1)} = 1 - \overline{P_D(S,I,N)} \qquad (11\text{-}32)$$

and

$$\epsilon_2 = \overline{P(1|0)} = \overline{P_D(S = 0,I,N)} \qquad (11\text{-}33)$$

where it is understood that $P(0|1)$ and $P(1|0)$ are actually the conditional probabilities $P\{(0|1)|(\xi,\mu)\}$ and $P\{(1|0)|\xi\}$. If we let the joint probability density of ξ and μ be $p_1(\xi,\mu)$, and the probability density of ξ alone be $p_2(\xi)$, then

$$\epsilon_1 = 1 - \int_{-1}^{1} \int_0^1 P_D(S,I,N)\, p_1(\xi,\mu)\, d\xi\, d\mu$$

$$= 1 - 2\int_{-1}^{1} \int_0^1 \int_{V_{Bn}}^{\infty} \rho \exp\left\{-(\rho^2 + S_n{}^2 + \xi^2 I_n{}^2 \right. \tag{11-34}$$

$$\left. + 2\mu\xi S_n I_n)\right\} I_o \left(2\rho\sqrt{S_n{}^2 + \xi^2 I_n{}^2 + 2\mu\xi S_n I_n}\right) p_1(\xi,\mu)\, d\rho\, d\xi\, d\mu$$

and

$$\epsilon_2 = \int_0^1 P_D(S=0,I,N)\, p_2(\xi)\, d\xi$$

$$= 2\int_0^1 \int_{V_{Bn}}^{\infty} \rho \exp\left\{-(\rho^2 + \xi^2 I_n{}^2)\right\} I_o (2\rho\xi I_n)\, p_2(\xi)\, d\rho\, d\xi \tag{11-35}$$

The increased complexity of equations (11-34) and (11-35) over equations (11-30) and (11-31) for individual members of the ensemble of error probabilities is obvious from the equations themselves. However, examination of the functional dependence of P_D on the random variables ξ and μ suggests a simplifying approximation which makes calculations of the error probabilities feasible. The random variable μ enters $P_D(A_S A_I,N)$ only in the cross product term of the expression for the envelope voltage E. Therefore, if either $R_S >> 1$ or $R_S << 1$, the maximum value of the cross product term is small relative to the total envelope voltage, so that

varying μ between its limits of -1 to +1 produces only a small variation in P_D. The occupation parameter ξ, on the other hand, varies from 0 to 1, and enters both the cross-product term and the interference power term of the envelope function. The effect of its variation may, therefore, be greater than that of μ, especially if A_I^2 is large compared to $(A_S^2 + N)$. However, it seems intuitively clear that, except in this latter case, the overall variation will still be relatively small.

Since the functional dependence of the error probabilities on the random variables μ and ξ is weak for most of the interference situations of interest, it becomes useful to consider the Taylor series expansions of the probability of detection about the mean values $\bar{\mu}$ and $\bar{\xi}$ of these two variables. This series can then be inserted into equations (11-34) and (11-35) and integrated term by term. The first term of the series is the constant term $P_D\{(A_S, A_I, N)|\bar{\xi}, \bar{\mu}\}$ (which is independent of $\bar{\mu}$ for S=0). The integrals of the linear terms vanish, because of the property of the average value of a random variable that

$$\int_{-\infty}^{\infty} (x - \bar{x})\, p(x)\, dx = 0 \tag{11-36}$$

Thus, from equation (11-34) we obtain

$$\begin{aligned} \epsilon_1 = \overline{P(0|1)} &= 1 - P_D\{(A_I,N)|\bar{\xi},\bar{\mu}\} \int_{-1}^{1}\int_{0}^{1} p_1(\xi,\mu)\, d\xi\, d\mu \\ &+ \int_{-1}^{1}\int_{0}^{1} T_n(\xi,\mu)\, p_1(\xi,\mu)\, d\xi\, d\mu \end{aligned} \tag{11-37}$$

and from equation (11-35) we obtain

$$\begin{aligned} \epsilon_2 = \overline{P(1|0)} &= P_D\{(S{=}0,I,N)|\bar{\xi}\} \int_{0}^{1} p_2(\xi)\, d\xi \\ &+ \int_{0}^{1} T_n(\xi)\, p_2(\xi)\, d\xi \end{aligned} \tag{11-38}$$

where T_n represents the sum of all the higher-order terms (second degree and higher) of the appropriate Taylor series. The weak dependence of the detection probability on ξ and μ implies that the contribution of the higher order terms may reasonably be neglected. Noting that the integrals multiplying the constant terms in equations (11-37) and (11-38) are, by definition, unity, the expected value of the error probabilities becomes

$$\epsilon_1 \doteq 1 - P_D\{(S,I,N)|\overline{\xi},\overline{\mu}\} \quad (11\text{-}39)$$

and

$$\epsilon_2 \doteq P_D\{(S=0,I,N)|\overline{\xi}\} \quad (11\text{-}40)$$

But, we can assume that the cross term phase angle, ϕ, is uniformly distributed over $[0, 2\pi]$, so that

$$\overline{\mu}(\equiv\overline{\cos\ \phi}) = 0 \quad (11\text{-}41)$$

Furthermore, $\overline{\xi}$ is nothing more than the average fraction of time that the interference pulses are present at the IF input. Hence,

$$\overline{\xi} = \delta \quad \text{(Interference Duty Cycle)} \quad (11\text{-}42)$$

Combining equations (11-39) through (11-42) with equations (11-30) and (11-31) we obtain the following results for ϵ_1 and ϵ_2:

$$\epsilon_1 = 1 - 2\int_{V_{Bn}}^{\infty} \rho \exp\{-(\rho^2 + S_n{}^2 + \delta^2 I_n)\} I_o(2\rho\sqrt{S_n{}^2 + (\delta I_n)^2})\, d\rho \quad (11\text{-}43)$$

and

$$\epsilon_2 = 2\int_{V_{Bn}}^{\infty} \rho \exp\{-\rho^2 - (\delta I_n)^2\}\ I_o\ (2\rho\delta I_n)\, d\rho \quad (11\text{-}44)$$

Before discussing the graphical representation of these re-

sults, we should make some assumption concerning the choice of the threshold voltage, V_B. In many digital systems, the decision threshold is defined by applying the Neyman-Pearson criterion, whereby the probable error of commission is specified at some value, ϵ_{20}, and V_B is chosen so as to yield this error probability when noise alone is present (S=I=0). Applying this criterion, equation (11-44) would yield

$$\epsilon_{20} = \int_{V_{Bn}}^{\infty} \rho \exp(-\rho^2)\, d\rho = \exp(-V_{Bn}^2) \qquad (11\text{-}45)$$

since $I_o(0) = 1$. Thus, V_{Bn} can be given by

$$V_{Bn} \equiv \frac{V_B}{\sqrt{2N}} = \ln\sqrt{\frac{1}{\epsilon_{20}}} \qquad (11\text{-}46)$$

Curves of the expected error probability, ϵ_1, are given in Figures 11-4, 11-5, 11-6, and 11-7, where each set of curves corresponds to a fixed value of V_{Bn} (or ϵ_{20}). The curves are plotted with respect to S_n, with δI_n as a parameter. Figure 11-7 gives curves of ϵ_2 vs. δI_n with V_{Bn} (or ϵ_{20}) as a parameter.

It should be pointed out that the interference duty cycle, δ, appears in equations (11-43) and (11-44) because it happens to be the average value of ξ. Its presence in these expressions as a seemingly deterministic quantity would be completely valid if the probability distribution for ξ was strongly concentrated near the average value $\bar{\xi} = \delta$ (i.e., if the occupation parameter for virtually every bit interval was close to δ). That this is not true for the case $\tau = T_B$ is merely a demonstration of the fact that equations (11-39) and (11-40) are indeed approximations which, to a first order, account for the statistical nature of ξ.

Let us now examine the case $\tau >> T_B$, characterized by long interval (compared to the signal bit interval) during which the interference is present as a cw sinusoid, followed by long intervals during which the interference is absent. We know that, in this case, the occupation parameter ξ for most signal bit intervals will be either 0 or 1; furthermore, that the value $\xi = 1$ will occur in a fraction δ of all bit intervals, while the value

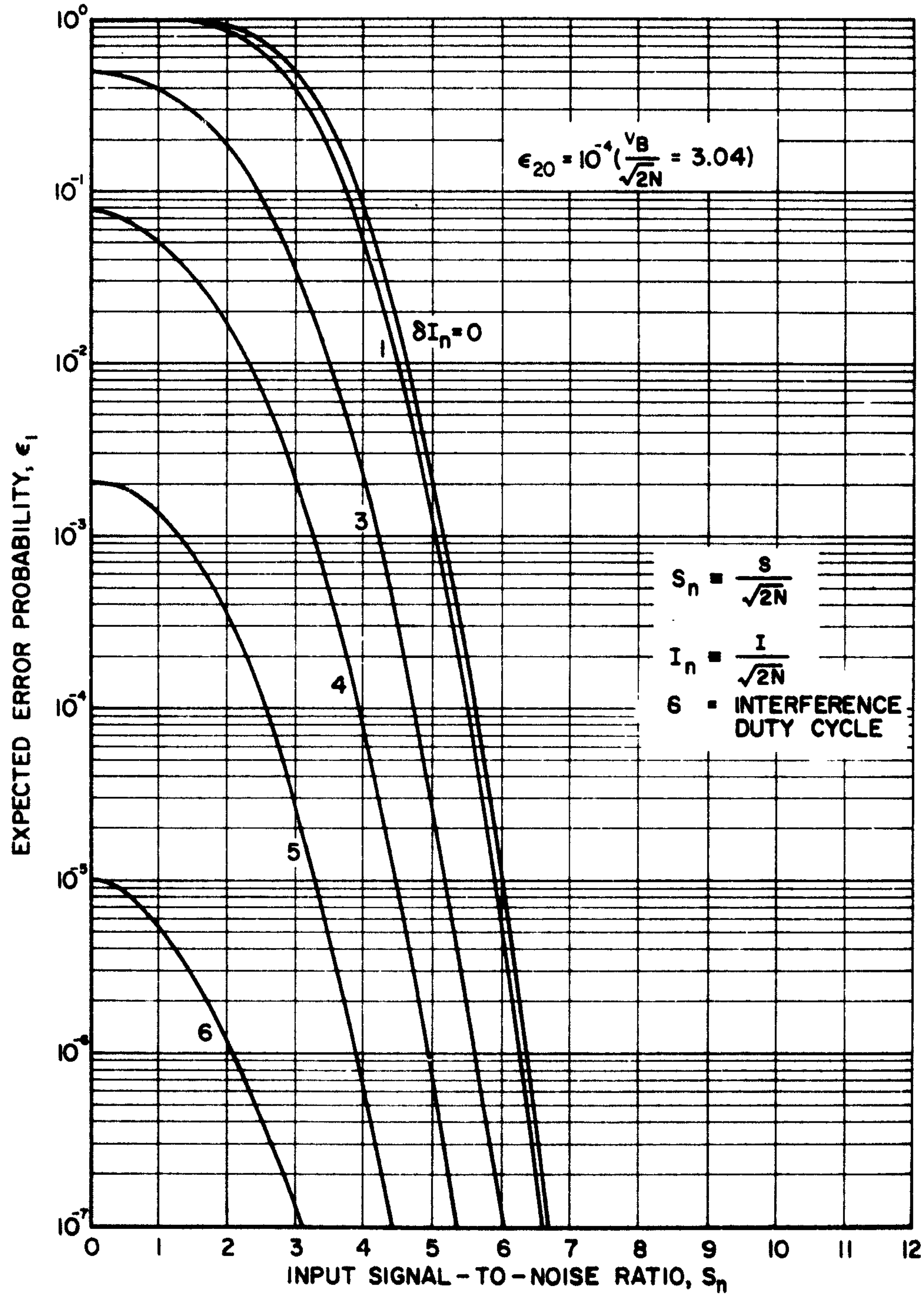


Figure 11-4. Expected Probability For Errors of Omission ($\epsilon_{20} = 10^{-4}$)

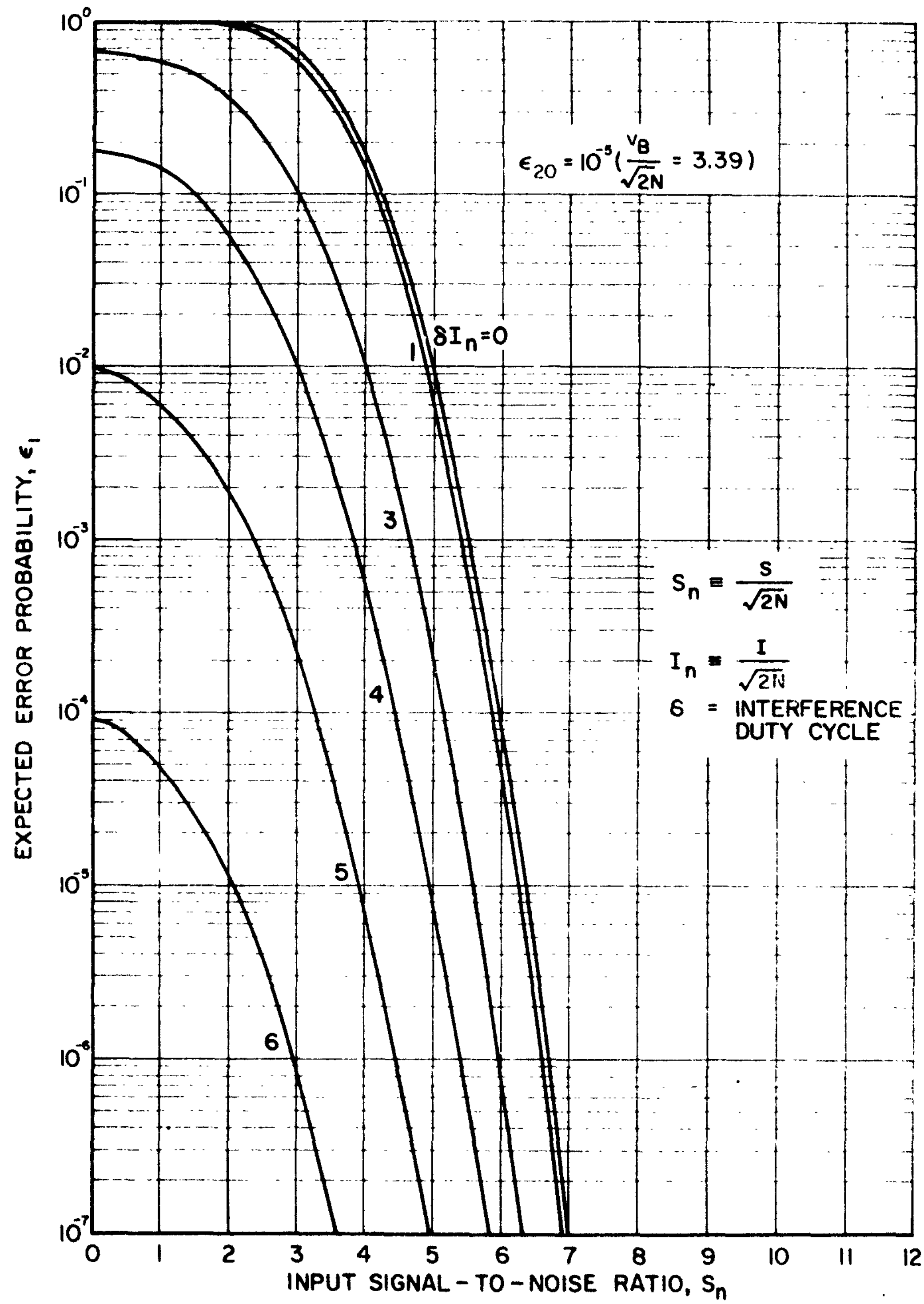


Figure 11-5 Expected Probability for Errors of Omission

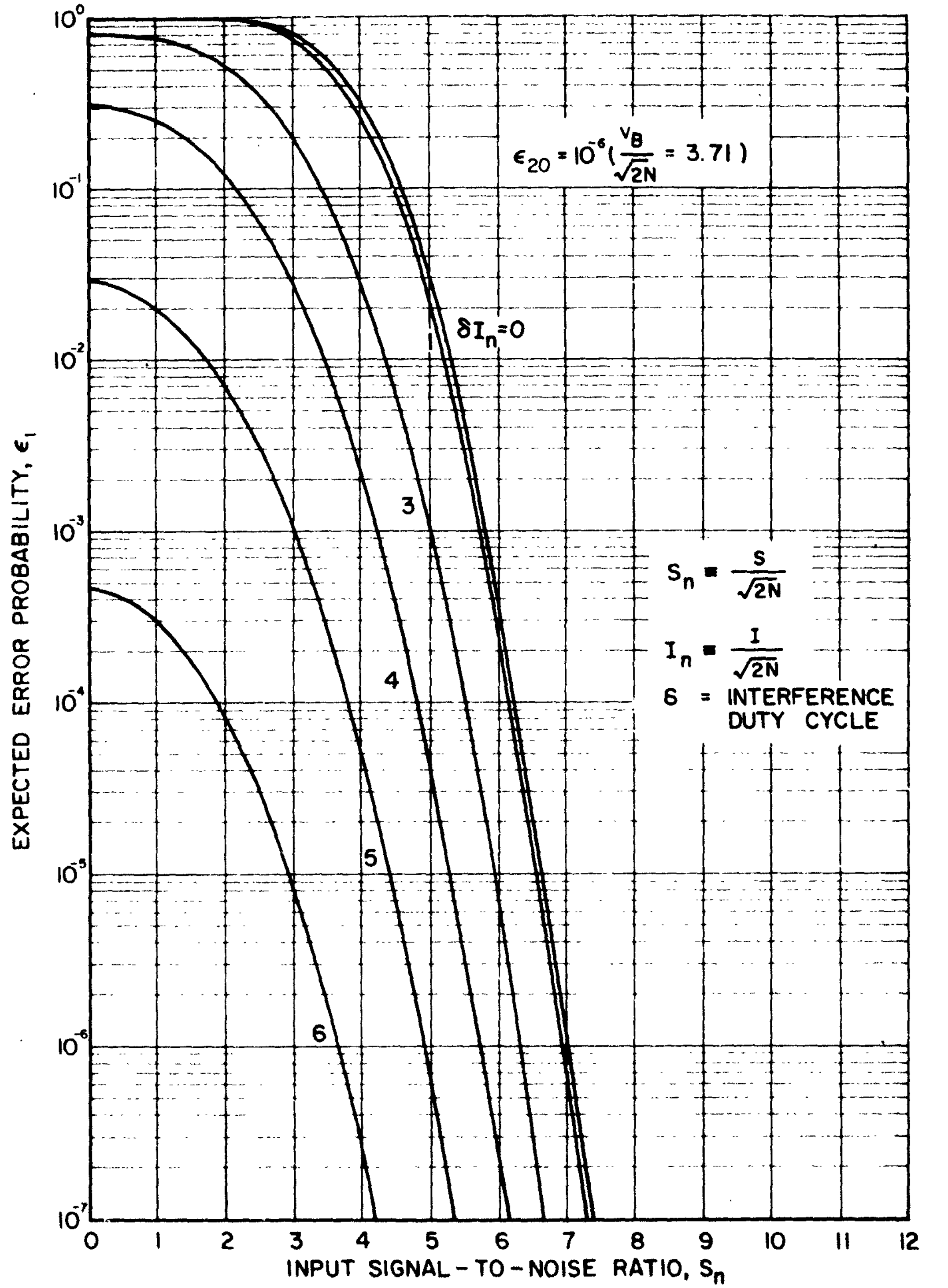


Figure 11-6 Expected Probability for Errors of Omission
$\epsilon_{20} = 10^{-6}$

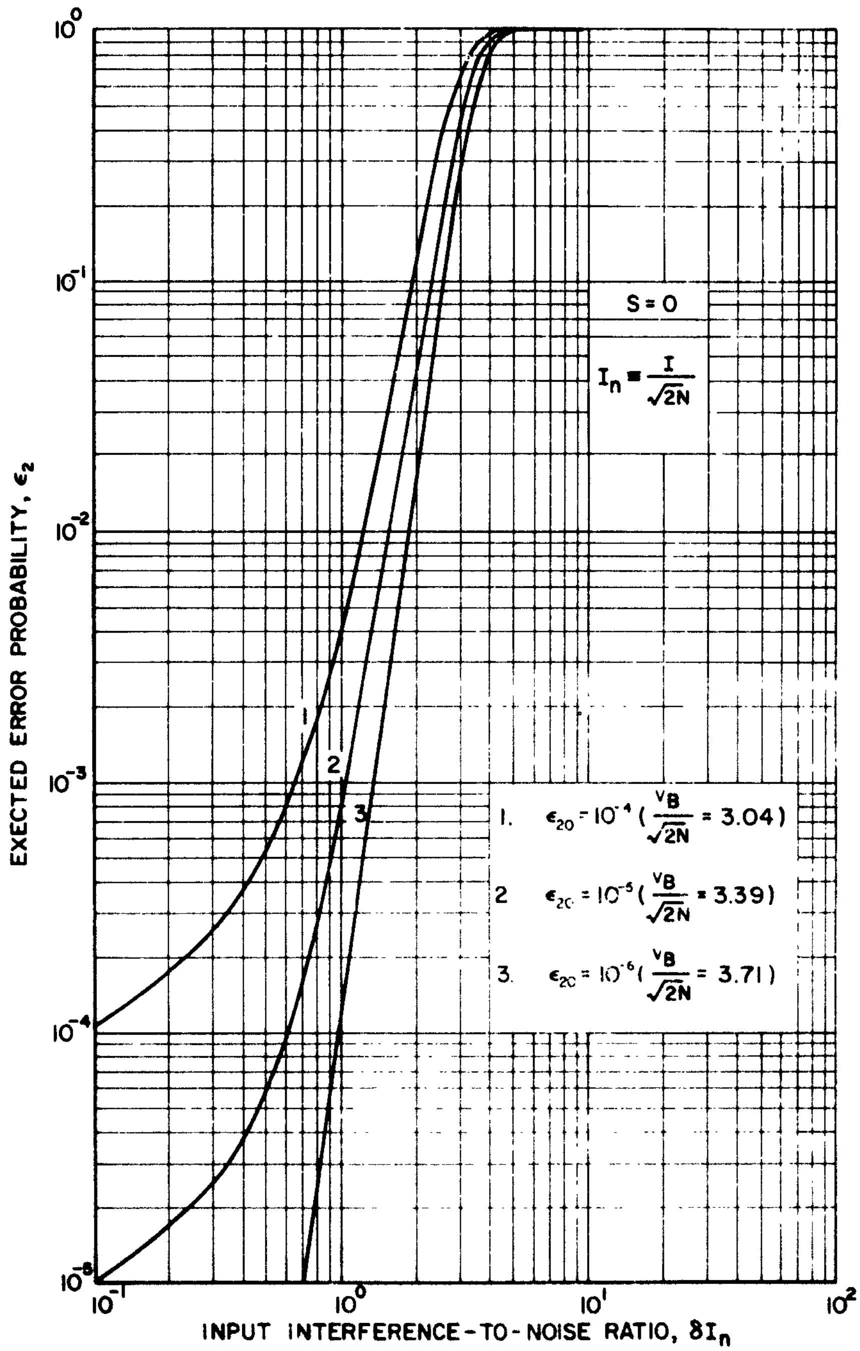


Figure 11-7. Expected Probability for Errors of Commission

$\xi = 0$ will occur in a fraction $(1 - \delta)$ of all intervals. Thus, during those intervals in which an interference pulse is present, ξ is a deterministic quantity equal to 1 so that, evaluating equations (11-39) and (11-40) accordingly, we obtain equations (11-43) and (11-44) with δ replaced by 1. Conversely, during those intervals in which no interference pulse is present, ξ is a deterministic quantity equal to 0 so that, evaluating equations (11-39) and (11-40) accordingly, we obtain equations (11-43) and (11-44) with δ replaced by 0. Over many pulse repetition periods of the interference signal, the average values of ε_1 and ε_2 will thus be given by

$$\overline{\varepsilon}_1 = \delta\cdot\varepsilon_1(\delta=1) + (1-\delta)\cdot\varepsilon_1(\delta=0) \tag{11-47}$$

and

$$\overline{\varepsilon}_2 = \delta\cdot\varepsilon_2(\delta=1) + (1-\delta)\cdot\varepsilon_2(\delta=0) \tag{11-48}$$

where ε_1 and ε_2, for the cases $\delta = 0$ and $\delta = 1$, can be evaluated using equations (11-43) and (11-44). By defining $\overline{\varepsilon}_1$ and $\overline{\varepsilon}_2$ in this way, we have directly accounted for the statistical nature of ξ, so that equations (11-39) and (11-40) are approximations only with respect to the statistics of μ. To this extent, the results obtained for the case $\tau >> T_B$ are inherently more accurate than those obtained for $\tau = T_B$.

DISCUSSION OF RESULTS

The major assumptions inherent in the results obtained above will now be summarized and discussed:

1. The analysis has been based upon a pulsed carrier receiver model which is believed to be suitable for a wide class of digital communication systems, with a correspondingly general applicability of the results.
2. We have assumed from the outset that the receiver is synchronized to the received sequence of desired signal bits, and have derived the corresponding error probabilities in the presence of pulse interference.
3. Several simplifying analytical approximations have been used, the most important of which are:
   a. The assumed linear buildup and decay of the IF output pulse due to a rectangular input pulse.

b. The attainment of the ensemble averages for $P(0|1)$ and $P(1|0)$ by neglecting the higher order terms in the Taylor series expansion of $P_D\{(S,I,N)|\xi,\mu\}$ about ξ and μ.

These approximations have both been made for the sake of analytical tractability, and generality and usability of the results obtained.

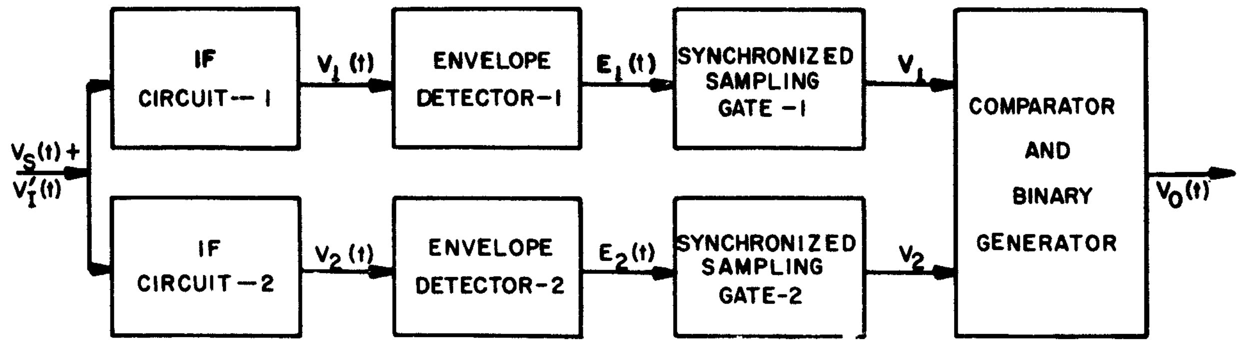
In this section the results have been obtained in terms of error probability. All that is necessary to complete a performance degradation problem is to transform the error probability to an appropriate form to the particular problem at hand.

SECTION 12

A DESIRED FREQUENCY SHIFT KEY RECEIVER INTERFERED WITH BY AN UNDESIRED PULSED SIGNAL

INTRODUCTION

The portion of the frequency shift key (FSK) receiver to be treated in the following analysis is depicted in Figure 12-1.



where

$v_S(t)$ = desired signal

$v_I'(t)$ = IF input undesired signal

$v_o(t)$ = low-pass filter output signal

$[v(t), E(t), V]_{1 \text{ or } 2}$ = the respective output of the IF, the envelope detector and the sampling gate for channel 1 or 2.

Figure 12-1. Frequency Shift Key Receiver

The assumptions used for the following receiver subsections and the input signals can be stated as follows:

1. The input desired signal is given by

$$v_S(t) = A_S \cos \{[\omega_1 + (\omega_2 - \omega_1)m_S(t)]t\} \tag{12-1}$$

where $\omega_2 > \omega_1$, by definition, and $m_S(t)$ is a random binary waveform which is either 0 or 1 at all times, but changes states only at integral multiples of T_B, i.e., at $t = T_B$ and/or $t = 2T_B$, and/or $t = 3T_B$, etc. Thus, $v_S(t)$ is at all times a fixed-amplitude sinusoid whose frequency is either f_1 or f_2, where the occurrences of $f = f_1$ and $f = f_2$ are synonomous with the occurrences of $m_S =$ 0 and $m_S = 1$, respectively.

2. The input interference signal is given by

$$v_I(t) = A_I m_I(t) \cos [(\omega_o + \Delta\omega)t + \theta_I] \tag{12-2}$$

where

$$\omega_o (\equiv 2\pi f_o) = \frac{1}{2}(\omega_1 + \omega_2) = \pi(f_1 + f_2) \tag{12-3}$$

and $m_I(t)$ is a binary waveform of unit amplitude. (Note that we have, in this case, defined the system center frequency, f_o, as the frequency midway between f_1 and f_2.) The duty cycle and average pulse width for the waveform $m_I(t)$ are denoted by δ and τ, respectively. We will consider here all possible values of δ and two special cases for τ, namely, $\tau = T_B$, and $\tau >> T_B$.

3. In addition to desired signal plus interference, the input is assumed to consist of white, gaussian receiver noise. The resultant narrowband gaussian noise voltage at the output of each IF circuit is assumed to have a mean square value N.

4. Each IF circuit is assumed to exhibit a second-order Butterworth bandpass filter response centered about the channel

frequency.* Such a filter is characterized by a maximally flat frequency response within the passband and a 40 db/decade slope outside the passband. Assuming that the two IF circuits have the same 3-db half-bandwidth, β_{IF}, we can write their voltage gains as

$$|H_1(jf)| = \frac{1}{\sqrt{1 + \left(\frac{f-f_1}{\beta_{IF}}\right)^4}} \tag{12-4}$$

and

$$|H_2(jf)| = \frac{1}{\sqrt{1 + \left(\frac{f-f_2}{\beta_{IF}}\right)^4}} \tag{12-5}$$

We assume further that the frequency separation between the two channels is equal to the full 3-db bandwidth, i.e.,

$$f_2 - f_1 = 2\beta_{IF} \tag{12-6}$$

Combining equations (12-6) and (12-3) with equations (12-4) and (12-5), we obtain the IF voltage gains in the form

$$|H_1(jf)| = \frac{1}{\sqrt{1 + \left(1 + \frac{\Delta f}{\beta_{IF}}\right)^4}} \equiv g_1(\Delta f) \tag{12-4a}$$

and

$$|H_2(jf)| = \frac{1}{\sqrt{1 + \left(1 - \frac{\Delta f}{\beta_{IF}}\right)^4}} \equiv g_2(\Delta f) \tag{12-5a}$$

---

*It is apparent that for future work other filter characteristics are also desired. The following discussion therefore serves as a guide to these calculations.

where

$$\Delta f \equiv (f - f_o) \qquad (12\text{-}7)$$

For this case, the rejection of signals at frequency f_1 by channel 2 (and vice versa) is given by

$$g_1(+\beta_{IF}) = g_2(-\beta_{IF}) = \frac{1}{\sqrt{17}} \qquad (12\text{-}8)$$

This corresponds to roughly 12 db of signal isolation between the two FSK channels. It can be seen from equation (12-4a) and (12-5a) that $g_1(\Delta f)$ and $g_2(\Delta f)$ are displaced equally and oppositely about the center frequency f_o, that is,

$$g_1(\Delta f) = g_2(-\Delta f) \qquad (12\text{-}9)$$

where the upper 3-db cutoff frequency of $g_1(\Delta f)$ and the lower 3-db cutoff frequency of $g_2(\Delta f)$, coincide with f_o. See Figure 12-2.

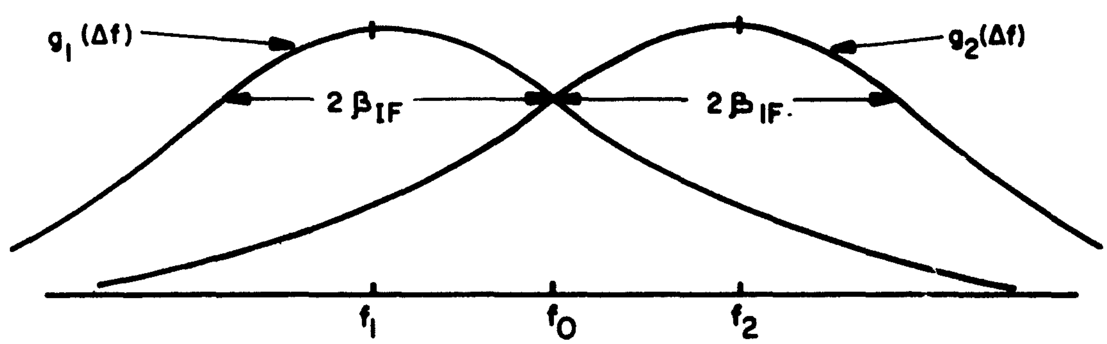


Figure 12-2. IF Voltage Gains

5. Another important assumption made about the IF circuits is that

$$\beta_{IF}T_B = 0.6 \qquad (12\text{-}10)$$

This is a typical design value for the circuits considered because it yields the maximum peak signal-to-rms-noise ratio for desired pulses. From an analytical standpoint, it permits us to approximate the pulse response of either IF circuit in the manner depicted in SECTION 11 by equation (11-4) and Figure 11-1, where $g(\Delta f)$ is the steady state response, $g_1(\Delta f)$ or $g_2(\Delta f)$, of the circuit being considered. Thus, the IF output pulse builds up linearly in time, T_B, to its steady-state value, and decays linearly at the same rate to zero, commencing with the trailing edge of the applied pulse.

6. The FSK system is assumed to be bit-synchronous, and the receiver is assumed to be in proper synchronization with the desired signal. Thus, at the end of each bit interval, the envelopes of the two IF outputs are sampled and compared. The receiver output for that interval is derived on the basis of the following rule: If the envelope sample from channel 1 exceeds that from channel 2, the receiver reacts as though the bit was transmitted at frequency f_1, and a "0" is generated at the receiver output; if the sample from channel 2 exceeds that from channel 1, the receiver reacts as though the bit was transmitted at frequency f_2, and a 1 is generated at the receiver output. Under ideal conditions, therefore, the receiver will regenerate the binary waveform $m_S(t)$ which controls the transmitted carrier frequency, equation (12-1). That is,

$$v_o(t) \propto m_S(t) \quad \text{(Ideal Conditions)} \qquad (12\text{-}11)$$

7. The detection scheme described above can be implemented by means of the envelope detectors, synchronized sampling gates, and logic circuitry indicated in Figure 12-1. The envelope-sampling time at the end of each bit interval is assumed to be quite small compared to the width, T_B, of the interval, and all circuits are assumed to perform ideally.

8. For the sake of literary simplicity, the following terminology will be employed in subsequent discussion:

a. The desired signal will be said to be in state 1 whenever its frequency is f_1, and to be in state 2 whenever its frequency is f_2.

b. The receiver will be said to have made a state decision 1 (corresponding to an apparent signal frequency f_1) in

any bit interval for which a 0 is generated at the receiver output, and a state decision 2 (corresponding to an apparent signal frequency f_2) in any bit interval for which a 1 is generated at the receiver output.

INDICES OF SYSTEM PERFORMANCE

Bit Error Probabilities. In order to assess the effects of pulse interference on the FSK receiver, we must obtain a quantitative expression for system performance in terms of the parameters which characterize the receiver and its input signals. For the digital FSK system described above, the most fundamental index of system performance is the bit error probability, from which a bit error rate may be derived. In this initial analysis it will be assumed that bit error correlations may be neglected. The higher order bit error correlations, while easily derivable from this analysis, are not really useful unless one considers in detail the specific codes being employed by the digital system; this facet of the problem is beyond the scope of this preliminary study.

Thus a quantitative expression for the system performance may be obtained in terms of the two (conditional) error probabilities:

$P(1|2)$ = the probability of a state decision 1 given a signal state 2, and

$P(2|1)$ = the probability of a state decision 2 given a signal state 1.

Now it must be considered that for any bit interval in which the signal state is 2, the conditional probability $P(1|2)$ will depend upon the interference state during that interval, that is, upon the amplitude, phase, and fractional time of occurrence of the interference during the interval; similarly for $P(2|1)$ when the signal state is 1. But, such interference parameters as relative phase and fractional time of occurrence within the interval are statistical quantities which vary from bit to bit. Hence, $P(1|2)$ and $P(2|1)$ are statistical quantities which must be averaged over all possible interference states in order to obtain the expected error probabilities per bit. We will refer to such an ensemble average for $P(1|2)$ as a type 1 expected error probability, denoted by ϵ_1, while the ensemble average of $P(2|1)$ will be referred to as a type 2 expected error probability, denoted by ϵ_2. Thus,

$$\epsilon_1 \equiv \overline{P(1|2)} \qquad (12\text{-}12)$$

and

$$\epsilon_2 \equiv \overline{P(2|1)} \qquad (12\text{-}13)$$

Bit Error Rate. The total bit error rate can be derived in terms of ϵ_1 and ϵ_2 as follows: For any given bit interval, wherein there is no a priori knowledge of the signal state, the average error probability will be

$$\epsilon_o = \overline{P(1|2)}\, P(2) + \overline{P(2|1)}\, P(1) \qquad (12\text{-}14)$$

where P(2) is the probability of a signal state 2 and P(1) is the probability of a signal state 1. Using ϵ_1 and ϵ_2 for the expected values of the conditional probabilities, and the fact that

$$P(1) + P(2) = 1$$

we obtain

$$\epsilon_o = [1 - P(1)]\epsilon_1 + P(1)\epsilon_2 \qquad (12\text{-}14a)$$

The average bit error rate is obtained by multiplying the average error probability per bit by the bit rate, i.e.,

$$r = \frac{\epsilon_o}{T_B} \qquad (12\text{-}15)$$

or using equations (12-10) and (12-14a)

$$r = \frac{5}{3}\beta_{IF}\{[1 - P(1)]\epsilon_1 + P(1)\epsilon_2\} \qquad (12\text{-}16)$$

For a typical FSK system, P(1) = P(2) = 1/2, so that

$$r = \frac{5}{6}\beta_{IF}(\epsilon_1 + \epsilon_2) \qquad (12\text{-}16a)$$

A possibly more general approach is to define an effective or weighted error probability per bit, i.e.,

$$\varepsilon_w \equiv (1 - w)\varepsilon_1 + w\varepsilon_2 \tag{12-17}$$

where the individual error probabilities are weighted differently either because one has an opportunity to occur more frequently than the other, or is intrinsically more important in terms of information loss, or for a combination of these two reasons. The corresponding weighted error rate is

$$r_w = \frac{5}{3}\beta_{IF}[(1 - w)\varepsilon_1 + w\varepsilon_2] \tag{12-18}$$

Again, a typical FSK system exhibits symmetry with respect to the occurrences and relative importances of the two signal states, so that in general, $w = 1/2$ and r_w is identical to equation (12-16a).

The conditional error probabilities $P(1|2)$ and $P(2|1)$ and their expected values are critically dependent upon the deterministic parameters of the problem, i.e., A_S, A_I, N, Δf, etc. As a result of the symmetry of the state channels about f_o (depicted by equation (12-9) and Figure 12-2) it can be seen that the error probabilities exhibit a symmetry with respect to Δf, that is, for a given set $\{A_S, A_I, N, \text{etc.}\}$,

$$\varepsilon_1(\Delta f) = \varepsilon_2(-\Delta f) \tag{12-19}$$

Consequently, a full evaluation of either one of these expected error probabilities is sufficient to obtain the desired description of system performance. The expected error probability $\varepsilon_1 = \overline{P(1|2)}$ will be studied below.

DERIVATION OF EXPECTED BIT ERROR PROBABILITY

The type 1 error probability $P(1|2)$ is the probability that, for a given bit interval, the sampled envelope in channel 1 will exceed the sampled envelope in channel 2 even though the signal frequency is indeed f_2. Let us denote the values of the sampled envelopes by V_1 and V_2, and their joint probability density function by $p(V_1,V_2)$, where

$$\int_0^\infty \int_0^\infty p(V_1,V_2)\, dV_1\, dV_2 = 1$$

The type 1 error probability is obtained by integrating $P(V_1,V_2)$ over that portion of the V_1-V_2 plane in which $V_1 \geq V_2$. Thus

$$P(1|2) = \int_0^\infty \int_{V_1}^\infty p(V_1,V_2)\, dV_2\, dV_1 = \int_0^\infty \int_0^{V_2} p(V_1,V_2)\, dV_1\, dV_2 \tag{12-20}$$

The derivation of the type 1 expected error probability, ϵ_1, will therefore be performed in three analytical steps:

1. The joint probability density function, $p(V_1,V_2)$ will be obtained by processing the input signal, interference, and noise voltages through the IF stages and envelope detectors of the two channels.
2. The error probability $P(1|2)$ will then be derived using equation (12-20).
3. An approximation will be developed for the ensemble average of $P(1|2)$ over all possible interference states.

The above steps will be performed for the case where the interference pulse width is equal to the system bit width, $\tau = T_B$. The results so obtained will then be readily extended to the case $\tau >> T_B$.

To begin, we will develop the relationships between the receiver input components (signal, interference, and noise) and the sample voltages, V_1 and V_2, of the state channels. Let us consider the bit interval from $t = 0$ to $t = T_B$, and assume that the desired signal during this interval is a sinusoid of amplitude A_S and carrier frequency f_2. Further, let us consider an input rectangular interference pulse of amplitude A_I, carrier frequency $f_o + \Delta f$, and width T_B, a portion ξT_B of which is contained within the specified bit interval. Thus, an input interference pulse of width T_B which begins either at $t = -(1 - \xi)T_B$ or at $t = +(1 - \xi)T_B$, will occupy a portion ξT_B of the interval $0 \leq t \leq T_B$, and therefore satisfy the above condition. Since such a pulse occupies a fraction ξ of the specified interval, we will hereafter refer to ξ as the interference occupation parameter.

Because of the linearity of the IF circuits, we can invoke the principle of superposition and give their outputs in the general form

$$v_1(t) = n_1(t) + \{S_1(t) + I_1(t)\} \qquad (12\text{-}21a)$$

$$v_2(t) = n_2(t) + \{S_2(t) + I_2(t)\} \qquad (12\text{-}21b)$$

where

$n_1(t)$ and $n_2(t)$ are the narrowband, gaussian noise voltages at the individual IF outputs, each of which has a mean-square value N;

$S_1(t)$ and $S_2(t)$ are the individual IF outputs due to the desired signal at the IF inputs; and

$I_1(t)$ and $I_2(t)$ are the individual IF outputs due to the interference signal at the IF inputs.

Note that, whereas the signal frequency is assumed to be f_2, there is still a finite signal component at the IF output of channel 1. For the system model described above, the signal component in channel 2 will be stronger by a factor of $\sqrt{17}$, while the relative strengths of the interference output components will depend upon $\Delta f/\beta_{IF}$, equations (12-4a) and (12-5a).

We will assume here that the IF noise outputs $n_1(t)$ and $n_2(t)$ are statistically uncorrelated. This assumption is justified by the fact that most of the spectral energy in each of these signals is contained within the 3-db passband of the corresponding IF circuit. But, the IF passband for channel 1 extends from $(f_o - 2\beta_{IF})$ to f_o, while the IF passband for channel 2 extends from f_o to $(f_o + 2\beta_{IF})$; therefore, $n_1(t)$ and $n_2(t)$ have most of their spectral energy in adjacent, but not overlapping, frequency bands. The assumption of statistical independence between $n_1(t)$ and $n_2(t)$ will be found to be of great analytical convenience.

The bracketed terms of equations (12-21a) and (12-21b) are deterministic signals obtained from the superposition of the input desired and interfering signals. The envelope of these signals will therefore fluctuate at a rate no greater than the order of $|\Delta f - \beta_{IF}|$, which is the difference frequency between the carriers of the desired and interfering signals. We can reasonably assume, then, that the sampling time in each channel is short compared

to the period of these envelope fluctuations, i.e., that the deterministic signal component $\{S(t) + I(t)\}$ at each IF output is virtually a fixed-amplitude sinusoid during the length of the sampling period. The amplitudes of these sinusoids in the two channels, as well as the mean square noise voltages in these channels, combine to determine the joint probability density function $p(V_1, V_2)$. With this in mind, we will now find the amplitude of $\{S(t) + I(t)\}$ for each channel at the sampling time $t = T_B$.

To find $S_1(t)$ we employ the model depicted by Figure 11-2. Thus, if the input signal is a pulse of carrier frequency f_2 $(\equiv f_o + \beta_{IF})$ and amplitude A_S, with its leading edge at $t = 0$, then $S_1(t)$ is given in the specified bit interval by

$$S_1(t) = A_S g_1(+\beta_{IF})\frac{t}{T_B} \cos (\omega_o t + 2\pi\beta_{IF} t + \phi_{S1}) \qquad (12\text{-}22)$$

$$0 \leq t \leq T_B$$

where we have elected to include the phase shift ϕ_{S1} of the signal as it passes through the IF circuit. Near the end of the bit interval (i.e., within the sampling period, which is short compared to T_B but long compared to the period of $\cos (\omega_o t)$) $S_1(t)$ is therefore given by

$$S_1(t) = A_S g_1(+\beta_{IF}) \cos (\omega_o t + 2\pi\beta_{IF} t + \phi_{S1}) \qquad (12\text{-}23)$$

$$T_B \gg (T_B - t) \gg 2\pi/\omega_o$$

In order to find $I_1(t)$, we must consider the two cases for which the interference occupation parameter is ξ:

1. If the interference pulse begins at $t = -(1 - \xi)T_B$, then from the model postulated, $I_1(t)$ will be

$$I_1(t) = A_I g_1(\Delta f)A(t) \cos [\omega_o t + (\Delta\omega t + \theta + \phi_{I1})] \qquad (12\text{-}24)$$

$$0 \leq t \leq T_B$$

where

$$A(t) \equiv \begin{cases} (1 - \xi) + \frac{t}{T_B} & \text{when } 0 \le t \le \xi T_B \\ (1 + \xi) - \frac{t}{T_B} & \text{when } \xi T_B \le t \le T_B \end{cases} \quad (12\text{-}25)$$

ϕ_{I1} is the phase shift of the interference signal, and $g_1(\Delta f)$ is given by equation (12-4b).

2. If the interference pulse begins at $t = +(1 - \xi)T_B$, then from the model postulated, $I_1(t)$ will be

$$I_1(t) = A_I g_1(\Delta f) B(t) \cos\left[\omega_o t + (\Delta\omega t + \theta + \phi_{I1})\right] \quad (12\text{-}26)$$

$$0 \le t \le T_B$$

where

$$B(t) \equiv \begin{cases} 0 & \text{when } 0 \le t \le (1 - \xi)T_B \\ \frac{t}{T_B} - (1 - \xi) & \text{when } (1 - \xi)T_B \le t \le T_B \end{cases} \quad (12\text{-}27)$$

Near the end of the bit interval, i.e., during the sampling period, we see that

$$A(T_B) = B(T_B) = \xi \quad (12\text{-}28)$$

Therefore, within this interval (which is long compared to the period of cos $(\omega_o t)$) we find that

$$I_1(t) = \xi A_1 g_1(\Delta f) \cos\left[\omega_o t + (\Delta\omega t + \theta + \phi_{I1})\right] \quad (12\text{-}29)$$

$$T_B >> (T_B - t) >> 2\pi/\omega_o$$

regardless of whether the input interference pulse begins at $t = -(1 - \xi)T_B$ or at $t = +(1 - \xi)T_B$. This convenient result is

a byproduct of our linear approximation to the buildup and decay of the IF output.

Combining equations (12-23) and (12-29), and assuming that the sampling period is short compared to the period of cos ($\Delta\omega t - 2\pi\beta_{IF}t$), we can accurately obtain the sum of $S_1(t)$ and $I_1(t)$ during this period by use of the law of cosines. Thus,

$$\{S_1(t) + I_1(t)\} = E_1 \cos(\omega_o t + \alpha) \tag{12-30}$$

$$\frac{2\pi}{|\Delta\omega - 2\pi\beta_{IF}|} >> (T_B - t) >> \frac{2\pi}{\omega_o}$$

where

$$E_1 \equiv \{[A_S g_1(+\beta_{IF})]^2 + [\xi A_I g_1(\Delta f)]^2 + 2\xi A_I A_S g_1(+\beta_{IF}) g_1(\Delta f) \cos \psi_1\}^{1/2} \tag{12-31a}$$

$$\psi_1 \equiv \left\{(\Delta\omega - 2\pi\beta_{IF})t + (\theta + \phi_{I1} - \phi_{S2})\right\} \tag{12-31b}$$

and α is a complicated function of the various signal parameters which is of no concern here.

We thus see that within the duration of the sampling period, the deterministic component of the IF output can be given as a fixed-amplitude sinusoid of magnitude E_1, defined by equation (12-31). When no noise is added to this signal $[n_1(t) = 0]$, E_1 is the value of the envelope observed by the sampled envelope and becomes a statistical variable whose probability density function can be found to be

$$P_1(V_1) = 2\left[\frac{V_1}{2N} \exp\left\{-\frac{(V_1{}^2 + E_1{}^2)}{2N}\right\} I_o\left(\frac{V_1 E_1}{N}\right)\right] \tag{12-32}$$

where $I_o(\ )$ is the modified Bessel function of zero order. It should be noted that, for a given bit interval, the probability density function for the sampled envelope V_1 depends on the values of the statistical quantitities ξ and cos ψ_1 for that interval.

We now direct our attention to the sampled envelope of channel 2 and find that an identical development obtains. The form of the result is identical to that given by equations (12-31) and (12-32), except that all subscripts 1 must be changed to 2. Thus, the probability density function for the sampled envelope in channel 2 in the same bit interval is

$$P_2(V_2) = 2\left[\frac{V_2}{2N} \exp\left\{-\frac{V_2{}^2 + E_2{}^2}{2N}\right\} I_o \left(\frac{V_2 E_2}{N}\right)\right] \quad (12\text{-}33)$$

where

$$E_2 \equiv \{[A_S g_2(+\beta_{IF})]^2 + [\xi A_I g_2(\Delta f)]^2 + 2\xi A_I A_S g_2(+\beta_{IF}) g_2(\Delta f) \cos \psi_2 \}^{1/2} \quad (12\text{-}34)$$

and

$$\psi_2 \equiv \left\{(\Delta\omega - 2\pi\beta_{IF})t + (\theta + \phi_{I2} - \phi_{S2})\right\} \quad (12\text{-}34a)$$

In order to now find the joint probability density function $p(V_1,V_2)$, we invoke the assumption of statistical independence between $n_1(t)$ and $n_2(t)$ (4). Thus, if we assume given values for ξ, $\cos\psi_1$, and $\cos \psi_2$, then, the randomness of V_1 and V_2 depend only upon the respective noise statistics, and we can conclude that V_1 and V_2 are statistically uncorrelated. That is, given the values of ξ, $\cos \psi_1$, and $\cos \psi_2$, the joint probability distribution for V_1 and V_2 is merely

$$p(V_1,V_2) = p_1(V_1)p_2(V_2) \quad (12\text{-}35)$$

where $p_1(V_1)$ and $p_2(V_2)$ can be obtained from equation (12-31) through (12-34). It would perhaps be more informative to write these functions as conditional probability distributions, i.e.,

$$p[V_1,V_2|(\xi,\mu_1,\mu_2)] = p_1[V_1|(\xi,\mu_1)]\ p_2[V_2|(\xi,\mu_2)] \quad (12\text{-}36)$$

where

$$\mu_1 \equiv \cos \psi_1 \quad (12\text{-}37)$$

and

$$\mu_2 \equiv \cos \psi_2 \tag{12-38}$$

This notation reinforces the notion that the joint distribution must eventually be averaged over the ensemble of all possible states $\{\xi,\mu_1,\mu_2\}$, in order to obtain the expected error probability. For convenience, we will use the notation of equation (12-35) and (12-36) interchangeably where, if equation (12-35) is used, equation (12-36) will be assumed to be implied.

We can now find $P(1|2)$ for a given bit interval by integrating $p(V_1,V_2)$ as indicated by equation (12-20). Using equation (12-32) and (12-33) we obtain

$$P(1|2) = \int_0^\infty \int_{V_1}^\infty \frac{V_1V_2}{N^2} \exp\left\{-\left(\frac{V_1{}^2+V_2{}^2+E_1{}^2+E_2{}^2}{2N}\right)\right\} I_o\left(\frac{V_1E_1}{N}\right) I_o\left(\frac{V_2E_2}{N}\right) dV_2 dV_1 \tag{12-39}$$

where E_1 and E_2 are given by equations (12-31) and (12-34), and it is understood, again, that $P(1|2)$ is a conditional probability $P\{(1|2)|(\xi,\mu_1,\mu_2)\}$.

We now wish to find the ensemble average of $P(1|2)$ over all states $\{\xi,\mu_1,\mu_2\}$ so as to obtain the expected bit error probability ϵ_1. If the joint probability distribution of ξ, μ_1, and μ_2 is $P_j(\xi,\mu_1,\mu_2)$, then ϵ_1 is obtained from

$$\epsilon_1 \equiv \overline{P(1|2)} = \int_0^1 \int_{-1}^1 \int_{-1}^1 P(1|2)p_j(\xi,\mu_1,\mu_2)\, d\mu_1\, d\mu_2\, d\xi \tag{12-40}$$

$$= \int_0^\infty \int_{V_1}^\infty \int_0^1 \int_{-1}^1 \int_{-1}^1 p_o(\mu_1,\mu_2,\xi,V_1,V_2)\, d\mu_1\, d\mu_2\, d\xi\, dV_2\, dV_1$$

where $p_o(\mu_1,\mu_2,\xi,V_1,V_2)$ is the joint probability density function for μ_1,μ_2,ξ,V_1, and V_2 in any given bit interval. The latter form has been introduced here in order to demonstrate the formidability of this general result. The inherent difficulty of the problem

is compounded by the fact that the statistical variables $\mu_1 \equiv \cos \psi_1$, and $\mu_2 \equiv \cos \psi_2$ are strongly correlated by their mutual dependence on the carrier frequency and relative phase of the interfering signal. Examination of equations (12-31) and (12-34) reveals that the statistical parameters μ_1 and μ_2 enter into the conditional probability $P(1|2)$ only through the cross terms in the law of cosines from which E_1 and E_2 are derived. For the conditions $S/I >> 1$ and $S/I << 1$, the maximum values of these cross terms are relatively small, so that the variations of μ_1 or μ_2 from -1 to +1 should induce relatively small variations in $P(1|2)$. The occupation parameter ξ, on the other hand, enters into both the cross terms and the A_I^2 terms of E_1 and E_2, so that as ξ varies from 0 to 1, the change in $P(1|2)$ may be more significant. However, it can be argued from an intuitive standpoint that this change will still be small in all cases, except where A_I^2 is large compared to $(A_S^2 + N)$ and $\Delta f \leq 2\beta_{IF}$.

In view of the fact that the functional dependence of $P(1|2)$ on μ_1, μ_2, and ξ is rather weak over most of the interference states of interest, it is useful to consider the three-dimensional Taylor series expansion of this probability function about the average values of μ_1, μ_2, and ξ. If such an expansion were obtained, inserted into equation (12-40), and integrated term by term, the contributions due to the linear terms of the Taylor series would vanish because of the fundamental property of average values that

$$\int_{-\infty}^{\infty} (x - \bar{x})p(x)\, dx = 0$$

Hence, we would obtain

$$\varepsilon_1 \equiv P(1|2) = P\{(1|2)|(\bar{\mu}_1,\bar{\mu}_2,\bar{\xi})\} \int_0^1 \int_{-1}^1 \int_{-1}^1 p_j(\mu_1,\mu_2,\xi)\, d\mu_1\, d\mu_2\, d\xi$$

$$+ \int_0^1 \int_{-1}^1 \int_{-1}^1 T_n(\mu_1,\mu_2,\xi)p_j(\mu_1,\mu_2,\xi)\, d\mu_1\, d\mu_2\, d\xi \qquad (12\text{-}41)$$

where T_n is the sum of all the higher-order terms of the Taylor series expansion. We will now make the approximation that these higher-order terms are small compared to the fixed and linear terms of this expansion, a result which arises from the above discussion on the changes of $P(1|2)$ with μ_1, and μ_2, and ξ. Furthermore, the triple integral of the first term of equation (12-41) is unity, by definition of the joint distribution $p_j(\mu_1,\mu_2,\xi)$, so that we obtain

$$\epsilon_1 = P\{(1|2)|(\bar{\mu}_1,\bar{\mu}_2,\bar{\xi})\} \tag{12-42}$$

But, we can assume that the cross term phase angles ψ_1 and ψ_2 are uniformly distributed over $\{0,2\pi\}$, so that

$$\overline{\mu_1}(\equiv\overline{\cos\ \psi_1}) = \bar{\mu}_2(\equiv\overline{\cos\ \psi_2}) = 0 \tag{12-43}$$

Furthermore, ξ is nothing more than the average fraction of time that the interference pulses are present at the IF inputs, hence,

$$\bar{\xi} = \delta \quad \text{(Interference Duty Cycle)} \tag{12-44}$$

Combining equations (12-42), and (12-44) with equations (12-31), and (12-34), and (12-39), we obtain the following result for ϵ_1:

$$\epsilon_1 = 4\int_0^\infty \int_y^\infty xy \exp\left[-(x^2+y^2+\sigma_1{}^2+\sigma_2{}^2)\right]I_o(2\sigma_1 x)I_o(2\sigma_2 y)\, dx\, dy \tag{12-45}$$

where

$$\sigma_1{}^2 \equiv \frac{1}{2N}\left[\frac{A_S{}^2}{17} + \frac{(\delta A_I)^2}{1+\left(1-\frac{\Delta f}{\beta_{IF}}\right)^4}\right] \tag{12-46}$$

$$\sigma_2{}^2 \equiv \frac{1}{2N}\left[A_S{}^2 + \frac{(\delta A_I)^2}{1+\left(1+\frac{\Delta f}{\beta_{IF}}\right)^4}\right] \tag{12-47}$$

and the appropriate expressions for $g_1(\Delta f)$, $g_1(+\beta_{IF})$, and $g_2(\Delta f)$, and $g_2(+\beta_{IF})$ have been inserted using equations (12-4a) and (12-5b).

Curves of ϵ_1 vs $(\Delta f/\beta_{IF})$ are given in Figures 12-3, 12-4, and 12-5, wherein the parameter used in normalized interference, $(5A_2/\sqrt{2N})$, and each figure corresponds to a particular normalized value of desired signal strength $(A_S/\sqrt{2N})$. For given normalized values of the interference and desired signal strengths, $\epsilon_2(\Delta f)$ can be found as $\epsilon_1(-\Delta f)$.

Let us now consider what happens when $\tau \gg T_B$. In this case, the interference is present or absent for long intervals. When it is present, then, over large numbers of successive bit intervals, it will appear that ξ is uniquely 1; when it is absent, then, over a large number of successive bit intervals, it will appear that ξ is uniquely 0. Over a large number of successive interference pulses, therefore, we will obtain an average value for ϵ_1 given by

$$\overline{\epsilon}_1 = \delta\epsilon_1(\overline{\xi} = 1) + (1 - \delta)\ \epsilon_1(\overline{\xi} = 0) \qquad (12\text{-}48)$$

Now in equations (12-46) and (12-47), the parameter δ appears before A_I because it happens to give the value of ξ for the case $\tau = T_B$. Let us, however, write these expressions in the general form indicated by equation (12-42), i.e.,

$$\sigma_1{}^2 = \frac{1}{2N}\left[\frac{A_S{}^2}{17} + \frac{(\overline{\xi}A_I)^2}{1 + \left(1 - \frac{\Delta f}{\beta_{IF}}\right)^4}\right] \qquad (12\text{-}46a)$$

and

$$\sigma_2{}^2 = \frac{1}{2N}\left[A_S{}^2 + \frac{(\overline{\xi}A_I)^2}{1 + \left(1 + \frac{\Delta f}{\beta_{IF}}\right)^4}\right] \qquad (12\text{-}47a)$$

so that for the case $\tau \gg T_B$, ϵ_1 can be obtained from these expressions in conjunction with equations (12-45) and (12-48).

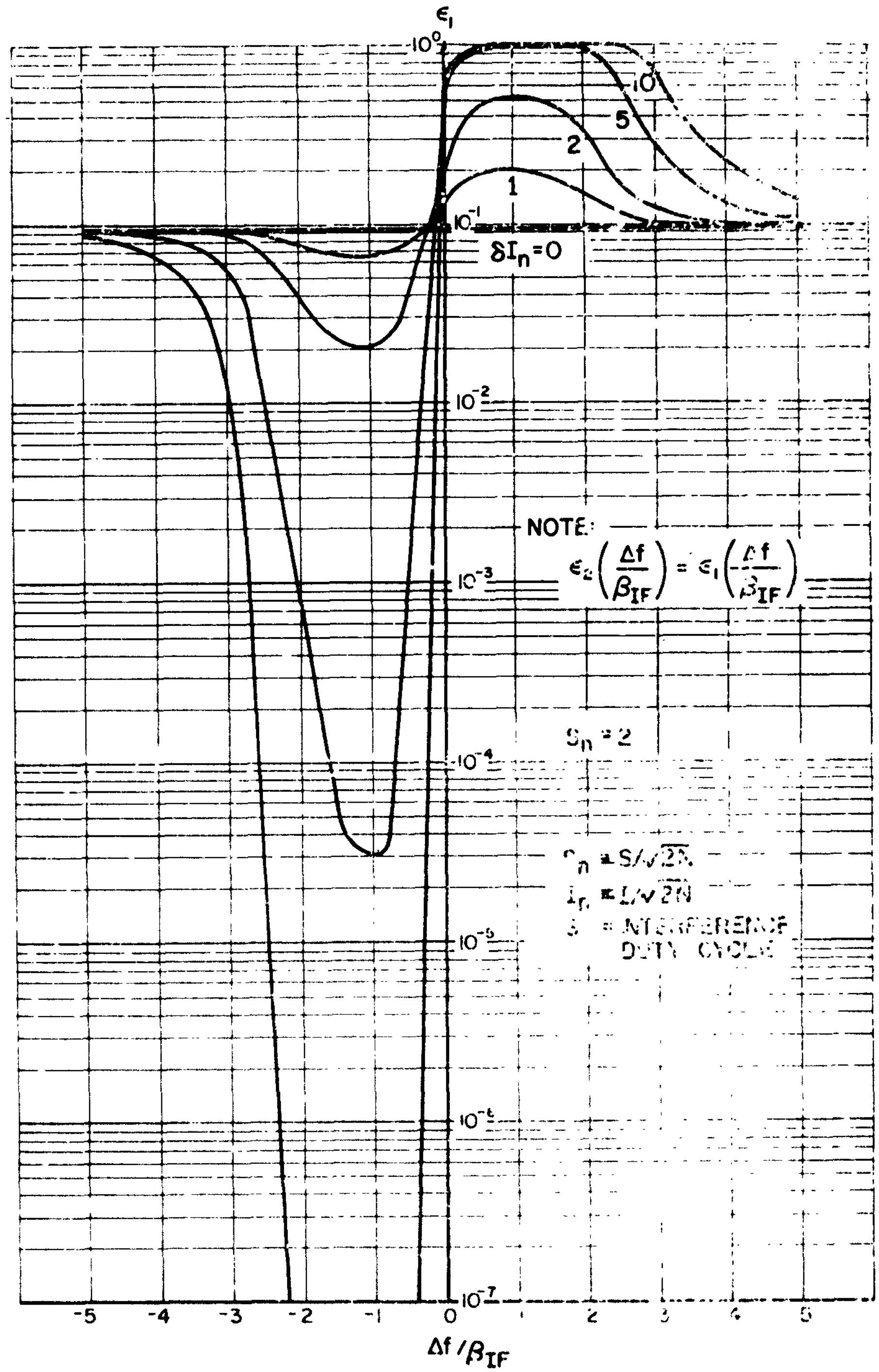


Figure 12-3. Expected Bit Error Probabilities; $S_n = 2$

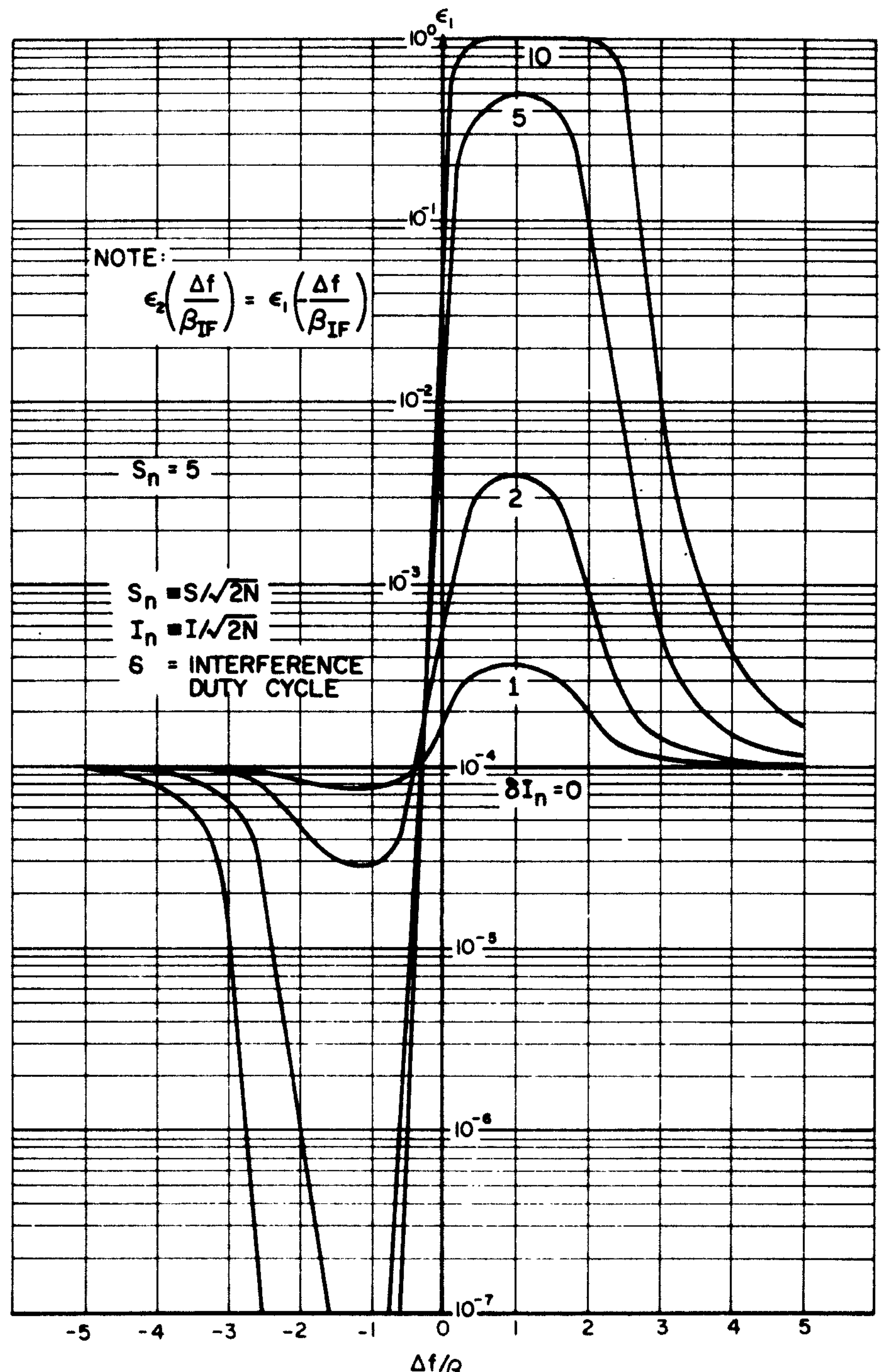


Figure 12-4 Expected Bit Error Probabilities; $S_n = 5$

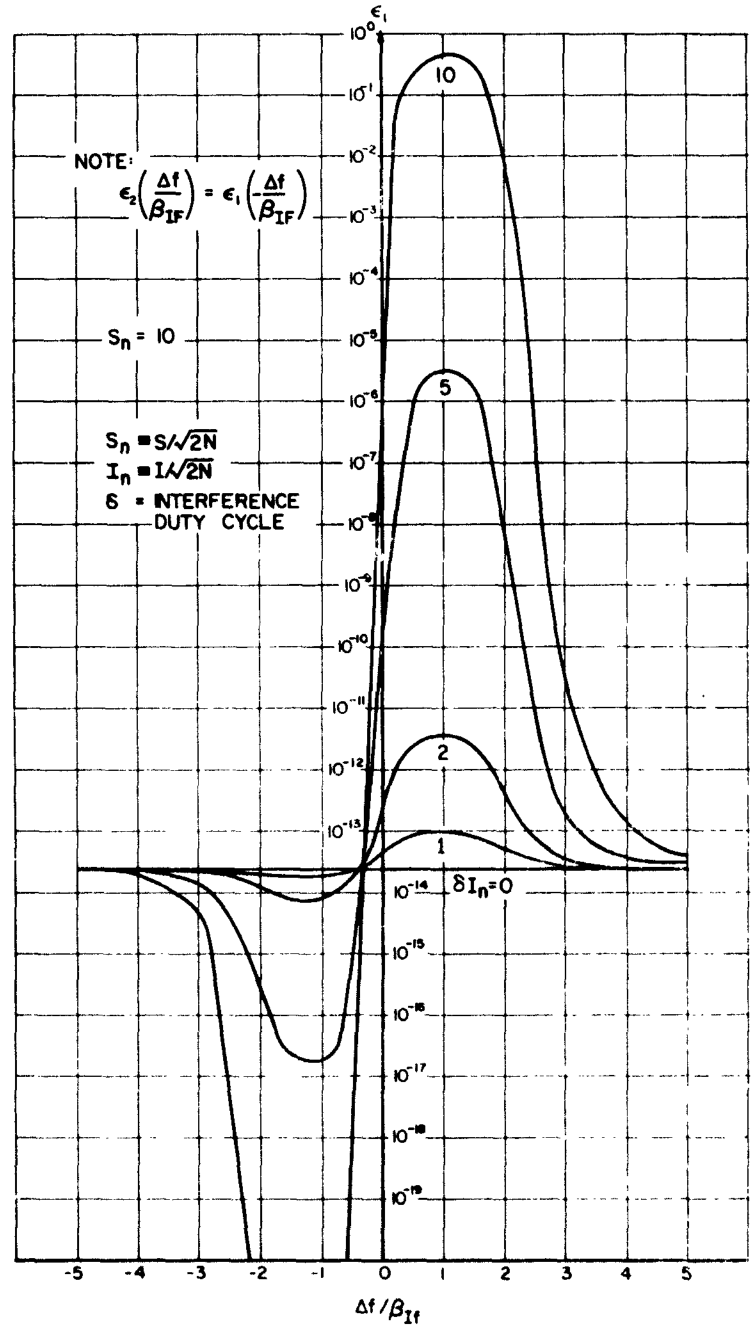


Figure 12-5. Expected Bit Error Probabilities; S_n = 10

It should be mentioned that there is no fundamental obstacle to preclude us from treating the case $\tau \gg T_B$ identically to the way we treated the case $\tau = T_B$, and obtaining the same results. The reason we do not is that such an approach invokes the tacit assumption that $\xi = \delta$ in each interval, i.e., that the probability density function for ξ is strongly concentrated about its mean value $\bar{\xi} = \delta$. For the case $\tau \gg T_B$, however, this is decidedly not so. Instead, the distribution for ξ is strongly concentrated near $\xi = 1$ and $\xi = 0$, so that, by using the approach, we have accounted for the true statistical distribution of ξ and have virtually eliminated any approximations regarding it. To this extent, the results obtained for $\tau \gg T_B$ are in general more accurate than those obtained for $\tau = T_B$.

DISCUSSION OF RESULTS

The major assumptions inherent in the results obtained above will now be summarized and discussed.

1. We have based the analysis on what is believed to be a typical FSK receiver in order to maximize the general applicability of the results.
2. We have assumed from the outset that the receiver is synchronized to the received sequence of desired signal bits, and have derived the corresponding error probabilities in the presence of pulse interference.
3. Several simplifying analytical approximations have been used, the most important of which are:
   a. The assumed linear buildup and decay of the IF output pulse due to a rectangular input pulse.
   b. The attainment of the ensemble averages of $P(1|2)$ and $P(2|1)$ by neglecting the higher order terms in the Taylor series expansion of $p_o(V_1,V_2,\mu_1,\mu_2,\xi)$ about $\bar{\mu}_1$, $\bar{\mu}_2$, and $\bar{\xi}$.

In this section the results have been obtained in terms of error probability. All that is necessary to complete a performance degradation problem is to transform the error probability to an appropriate form to the particular problem at hand.

SECTION 13

A DESIRED SIGNAL INTERFERED WITH BY RANDOM NOISE AND A DETERMINISTIC UNDESIRED SIGNAL

INTRODUCTION

The purpose of this section is to develop limiting restrictions on the general analysis of random noise signals in combination with deterministic desired and undesired signals. Since considerable analysis has already been performed on noise analysis problems, it is sufficient for this report to refer to the noise analysis problems of Middletown (5) or Rice (2) for the basic equation development. The equations, after being referenced to these works, will then be modified to account for an undesired interfering signal. In all cases the desired result is the highest degree of mathematical information that can be preserved, that is the nth order probability density function. For the case of gaussian noise this is the second order density function. Since in many cases the filtered low-pass output is desired, the power spectrum and/or autocorrelation function is next obtained from the density function. The filtered output can then be operated on to obtain the first and second moments of the output signal.

A flow diagram outlining the general approach to noise analysis problems is shown in Figure 13-1. The class of problems of particular interest are narrowband problems, that is, those problems in which the center frequency, ω_0, of the signal is much larger than the bandwidth of the signal centered about ω_0. The input noise at IF is in all cases to be considered gaussian. The IF portion of Figure 3-1 was not included in Figure 13-1 since this was not consistent with the basic form of the derivations in reference (5). However, the basic technique of handling off-tuned signals as discussed in APPENDIX III still applies here. The nonlinear detectors are to be investigated for AM, phase and FM detection types. The following section discusses the detailed development of noise analysis problems.

A DESIRED AM SIGNAL INTERFERED WITH BY RANDOM NOISE AND A DETERMINISTIC INTERFERING SIGNAL

The amplitude modulation detection problem that is to be discussed here consists of the analysis of narrow band gaussian noise added to a deterministic narrow band amplitude and phase modulated signal. The narrow band deterministic signal is produced by the addition of a desired signal and an off-tuned undesired signal of any modulation type. The general topic of linear AM detectors

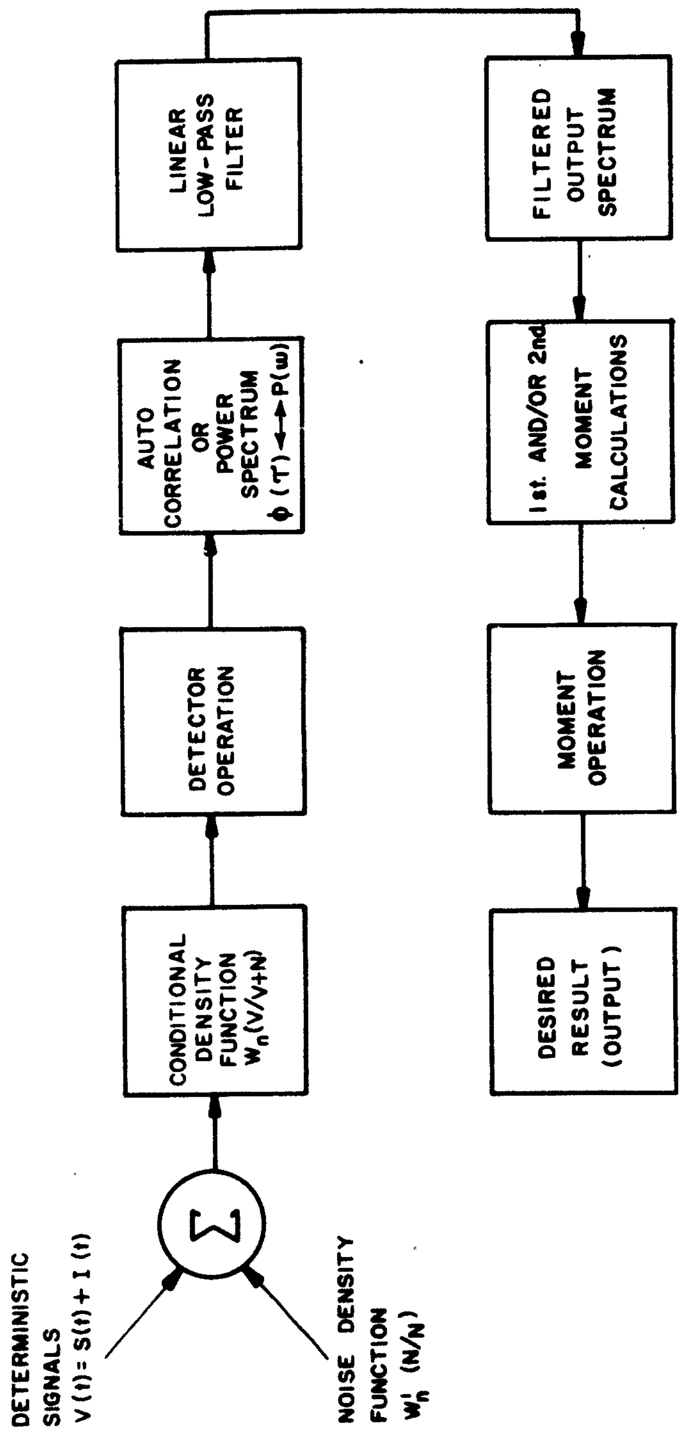


Figure 13-1. Random Noise Analysis Procedure

is discussed in reference (5), Chapter 13. The basic problem is to modify these results to the analysis of interference problems.

The starting point of this analysis is to determine the low-pass or zero zonal detector output covariance function. Only this is required, since the input noise is gaussian and is completely described by its second order probability density function. In order to determine the covariance function it is desired to compute the average of the combined input signal and therefore obtain

$$\overline{V'(t)V'(t+\tau)} = \overline{N(t)N(t+\tau)} + \overline{V(t)V(t+\tau)} + \overline{N(t)V(t+\tau)} \qquad (13\text{-}1)$$

where

$$V'(t) = N(t) + V(t) = N(t) + S(t) + I(t) \qquad (13\text{-}2)$$

$$V(t) = S(t) + I(t) \qquad (13\text{-}3)$$

$\overline{(\quad)}$ symbolizes the time average computation

Only the zero zonal auto and cross covariance functions are desired. These are symbolized from the reference (5), equation (13.46) as

$$M_Z(t)_o = K_Z(t)_o\Big|_{nxn} + K_Z(t)_o\Big|_{sxn} + M_Z(t)_o\Big|_{sxs+dc} \qquad (13\text{-}4)$$

where

$M_Z(t)_o$ = the zero zone (or low-pass filter) detector output covariance function.

$K_Z(t)_o\Big|_{nxn}$ = zero zonal detector output due to the auto product of noise with noise.

$K_Z(t)_o\Big|_{sxn}$ = zero zonal detector output due to the cross product of signal with noise.

$M_Z(t)_o\Big|_{sxs+dc}$ = zero zonal detector output due to the auto product of signal with signal.

These outputs have been generally solved for, and for a linear detector are given by equations (13.47a,b; 13.48) in reference (5).

$$K_Z(t)_o\Big|_{nxn} = \frac{\beta^2\Gamma(4)\psi}{8}\sum_{q=1}^{\infty}\frac{K_o(t)^{2q}\left\langle {}_1F_1(q-1/2;1;-P_1)\,{}_1F_1(q-1/2;1;-P_2)\right\rangle}{(q!)^2\Gamma[3/2-q]^2} \tag{13-5}$$

$$K_Z(t)_o\Big|_{sxn} = \frac{\beta^2\Gamma(4)\psi}{8}\sum_{m=1}^{\infty}\sum_{q=0}^{\infty} 2K_o(t)^{m+2q}\left\langle (P_1P_2)^{m/2}\,{}_1F_1(m+q-1/2;m+1;-P_1) \right. \tag{13-6}$$

$$\times\ \frac{\left. {}_1F_1(m+q-1/2;m+1;-P_2)\right\rangle}{q!\,(q+m)!\,(m!)^2\Gamma(3/2-m-q)^2}$$

$$M_Z(t)_o\Big|_{sxs+dc} = \frac{\beta^2\Gamma(4)\psi}{8\Gamma(q/4)}\left\langle {}_1F_1(-1/2;1;-P_1)\,{}_1F_1(-1/2;1;-P_2)\right\rangle \tag{13-7}$$

where

β = dynamic transconductance of a linear rectifier.

$\Gamma(\ )$ = gamma function.

$K_o(t)$ = envelope of the covariance function of a narrow band process.

${}_1F_1(\ )$ = hypergeometric function.

$\langle\ \ \rangle$ symbolizes a statistical average.

At this point the problem is solved, providing the statistical averages indicated by the above equations can be evaluated. However, for these results to be used in a realistic sense, the previous expressions must be greatly simplified for evaluation. In particular, the statistical averages involving the hypergeometric functions ${}_1F_1(\alpha,\beta,z)$ must be simplified by considering a reduced expression for $P_{1\ 2}$. This term, as a function of a general modulated interfering signal, is readily found to be (see reference (5), equation 13.21).

$$P(t) = a_\delta^2 V'^2(t) = V^2(t) \tag{13-8}$$

$$V^2(t) = A_S^2(t) + A_I^2(t) + 2A_S(t)A_I(t)\cos\left[\Delta\omega t + \theta_I + \phi_I(t)\right] \tag{13-9}$$

The only detailed problem that the reference obtains solutions for is that of a desired tone modulated signal given by

$$S(t) = A_S(1 + m_S \cos \omega_S t)\cos \omega_o t \tag{13-10}$$

In order to solve the general noise problem for this specialized signal the hypergeometric function of equation (13-6) is replaced by a three-term fit. The resulting approximation [reference (5), equation 13.61] is

$$S_a = \frac{|S_{\omega S}|}{\sqrt{2}} \approx \frac{\beta\psi^{1/2}}{\sqrt{4\pi}}\left[1 - 0.2289 A_S\left(1 + \frac{m_S{}^2}{4}\right)\right] \tag{13-11}$$

where

$$A_S{}^2(1 + m_S{}^2) \leq 10$$

and

$$M_Z(t)\Big|_{sxs} = (S_a{}^2/2)\cos \omega_S t$$

Since the solution to this equation is excessively involved, even for tone modulation, no solution to a more difficult problem will be attempted. This answer can, however, be used for certain specialized interfering problems. In particular the problem of an off-tuned carrier in conjunction with a desired carrier can be handled. For this case V(t) is given by

$$V^2(t) = A_S{}^2 + A_I{}^2 + 2A_S A_I \cos(\Delta\omega t + \theta_I) \tag{13-12a}$$

$$= (A_S{}^2 + A_I{}^2)\left[1 + \left(\frac{2A_S A_I}{A_S{}^2 + A_I{}^2}\right)\cos(\Delta\omega t + \theta_I)\right] \tag{13-12b}$$

This is of the same form as the previous desired signal and can therefore be used. Another form that can be used is that of an off-tuned carrier against noise. The answer to this problem will not be discussed here, but is given by equation (13.8-1,2) of reference (5).

The previous cases that have been discussed have been analyzed for a reasonably general range of signal-to-noise ratio. Restricting the range of signal-to-noise ratio to large and small values allows solutions of a more general form of the modulated interfering signal. The next portion discusses this approach.

<u>Large Signal Plus Interference-to-Noise Ratio Approximation</u>. In order to reduce the complexity of the previously discussed equations the confluent hypergeometric function will be approximated for large and small signal plus interference-to-noise ratios. For small values of Z the hypergeometric function can be approximated by

$$ {}_1F_1(\alpha,\beta,z) \approx 1 + \left(\frac{\alpha}{\beta}\right)z \tag{13-13}$$

For large values of Z the asymptotic expansion can be used and the series approximated by

$$ {}_1F_1(\alpha,\beta,z) \approx \frac{\Gamma(\beta)z^{-\alpha}}{\Gamma(\beta-\alpha)}\left[1 + \frac{\alpha(\alpha-\beta+1)}{z} + \dots\right] \tag{13-14}$$

Substituting these expressions into the basic equation (13-5) we obtain for the <u>weak signal case</u>

$$ K_Z(t)_o\Big|_{\substack{n x n\\ S+I>>n}} \approx \frac{\beta^2\Gamma(4)\psi}{8}\,\frac{K_o{}^2(t)}{\Gamma(1/4)}\left[1 + \frac{1}{2}(\overline{P}_1 + \overline{P}_2) + \frac{1}{4}(\overline{P_1P_2})\right] \tag{13-15}$$

$$ K_Z(t)_o\Big|_{\substack{s x n\\ S+I>>n}} \approx \frac{\beta^2\Gamma(4)\psi}{8}\,\frac{2K_o(t)}{\Gamma(1/4)}\left[\overline{(P_1P_2)^{1/2}} + \frac{1}{4}\overline{(P_1P_2)^{3/2}} + \frac{1}{2}\overline{(P_1^3P_2)^{1/2}}\right.$$

$$ \left. + \frac{1}{4}\overline{(P_2^3P_1)^{1/2}}\right] \tag{13-16}$$

$$ M_Z(t)_o\Big|_{s x s+dc} \approx \frac{\beta^2\Gamma(4)\psi}{8\Gamma(1/4)}\left[1 - \frac{1}{2}(\overline{P}_1 + \overline{P}_2) + \frac{1}{4}(\overline{P_1P_2})\right] \tag{13-17}$$

These quantities can in turn be expanded in terms of the desired and undesired signals as was given in equation (13-12) to obtain, for the n x n and s x s + dc case,

$$\overline{P}_1 = \overline{P}_2 = \overline{A_S^2(t)} + \overline{A_I^2(t)} = A_S^2 + A_I^2 \tag{13-18}$$

$$\begin{aligned}\overline{P_1P_2} = {} & \overline{A_S^2(t_1)A_S^2(t_2)} + \overline{A_I^2(t_1)A_I^2(t_2)} \\ & + \overline{2A_S(t_1)A_I(t_1)A_S(t_2)A_I(t_2)} + \overline{A_S^2(t_1)A_I^2(t_2)}\end{aligned} \tag{13-19}$$

The s x n terms can be obtained in a similar fashion. The algebra is slightly involved and will not presently be included. It should be noted, however, that these are just the terms that are usually neglected and consequently should be small. It should also be noted that equation (13-18) does not contain the time varying portion. Equation (13-19) does contain the time varying portion and essentially is the autocorrelation function of a square-law detected output.

For the large signal-plus-interference-to-noise ratio approximation, the hypergeometric approximation for the s x s + dc term after substituting the specific values of equation (13-7) becomes

$${}_1F_1(-1/2,1,-P_1) \approx \frac{2}{\sqrt{\pi}}\sqrt{-P_1} + \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{-P_1}} \quad . \tag{13-20}$$

After the appropriate operation of equation (13-7) on equation (13-20) the first term yields the first order ideal linear detector output, with the remaining terms being the secondary correction factors. [See equation (I-4) for the ideal model.]

The first term of the n x n autocovariance can be seen from equation (13-5) to be varying directly with the square of the input autocovariance (i.e., $K_o^2(t)$). The first term of the s x n autocovariance can be seen from equation (13-6) to be varying linearly with the input autocovariance (i.e. $K_o(t)$). For both of these cases the terms $K_o^2(t)$ or $K_o(t)$ must still be multiplied by a function, which requires taking the statistical average of a function which is a secondary function of the original input signal, V(t). Therefore, although the first $K_o^{1,2}(t)$ terms are of the same form as the small signal case, the total covariance function has become far more complicated due to the lack of signal suppression that was previously obtained for the small signal case.

Due to this complicated structure the output signal will be left in this form where the appropriate substitution for a particular problem should somewhat simplify the end result.

DISCUSSION OF RESULTS

The problems that have been discussed thus far in this section attempt to obtain the output covariance function for certain restricted problems of a general signal-to-noise ratio and general modulation conditions for the restricted cases of large and small signal-plus-interference-to-noise ratio. Although the output covariance functions have not been completed in all cases, the results have been left in a form that can be readily completed for a specific interference problem.

The Fourier transform can then, ideally, be taken of the covariance function to obtain the output low-pass power spectrum. The low-pass filter transfer function is then used in conjunction with this power spectrum to obtain the filtered output power spectrum. From this output power spectrum the first and second moments are readily obtained. From these outputs appropriate desired-to-undesired signal ratios can be calculated.

DESIRED PHASE MODULATED SIGNAL INTERFERED WITH BY RANDOM NOISE AND A DETERMINISTIC INTERFERING SIGNAL

The phase modulated detection problem that is to be discussed here consists of the analysis of narrow band gaussian noise added to a deterministic narrow band amplitude and phase modulated signal. The narrow band deterministic signal is produced by the addition of a desired signal and an off-tuned undesired signal of any modulation type.

The analysis of phase modulation is extremely similar to that of frequency modulation. There are also many analysis problems already existing for the FM problem. For this reason this section will use only a well known Fourier transform property in order to relate the results of this section to that of SECTION 9.

The purpose of this portion of this section is to obtain the output autocovariance function or its transform the power spectrum. What is desired, then, is the relationship between the FM power spectrum and the phase-modulated power spectrum. It can be readily shown that [as an example see reference (6) equation (6-201)]

$$\frac{d^n}{dt}[f(t)] \leftrightarrow (j\omega)^n F(\omega) \tag{13-21}$$

where

$F(\omega)$ is the Fourier transform of $f(t)$, this is symbolized by $f(t) \leftrightarrow F(\omega)$.

Since the instantaneous frequency is given by

$$f_{inst} = \frac{1}{2\pi} \frac{d}{dt}[\phi(t)] \tag{13-22}$$

and the power spectrum is given by

$$P(\omega) \underset{\lim T\to\infty}{=} \frac{2}{T}|F(\omega)|^2 \tag{13-23}$$

it follows that

$$P_{phase}(\omega) \equiv P_\phi(\omega) = \frac{P_{FM}(\omega)}{\omega^2} \tag{13-24}$$

It is therefore only necessary to obtain the FM power spectrum and divide by ω^2 to obtain the phase modulated spectrum. It is apparent that the frequency modulated spectrum could also be obtained from the phase-modulated spectrum by expression

$$P_{FM}(\omega) = \omega^2 P_\phi(\omega) \tag{13-25}$$

DESIRED FM SIGNAL INTERFERED WITH BY RANDOM NOISE AND A DETERMINISTIC INTERFERING SIGNAL

The FM detection problem that is to be briefly discussed here consists of the analysis of narrow band gaussian noise added to a deterministic narrow band amplitude and phase-modulated signal. The narrow band deterministic signal is produced by the addition of a desired signal and an off-tuned undesired signal of any modulation type. The general topic of an ideal FM detector is discussed in reference (5), chapter 14. The basic problem is to attempt to modify these results for the analysis of interference problems.

The extension from the case of AM and noise to the case of AM with noise and interference was relatively straightforward, although somewhat involved. The analysis of the FM case for the same approach is not as straightforward. The general amplitude level case appears excessively involved due to the multitude of cross product terms produced by the FM detection process. Only for the restricted case of a large or small desired-carrier-to-noise-plus-interference does the resulting solution reduce to a somewhat more reasonable level. However, even for these cases,

the answers are considerably involved. For the case of a sufficiently strong carrier the limiting condition discussed in APPENDIX II, equation (II-22) is a sufficient description of the noise spectrum. For a more detailed description it is apparent from previous discussions that what is also needed is a description of the power spectrums of the deterministic signal and interference. For the large carrier case this can be obtained from APPENDIX I, equation (I-186) and (I-188). The resulting output spectrum is then approximately obtained by the combination of the spectra obtained from these signals. When the signal and interference are smaller than the noise or, at most, the same order of magnitude, a different approach must be used. The pertinent autocovariance function is given by reference (5), equation (15.79b) as

$$K_o(t) = \frac{16\beta_F{}^2 K^2 r_o{}^2 b_o}{\pi^2} \{M_o(t) + a_o{}^2 M_1(t) + a_o{}^4[M_2(t) + 2\langle\dot{\phi}_1\dot{\phi}_2\rangle + O(a_o{}^6)]\} \tag{13-26}$$

where the pertinent terms are

$a_o{}^2$ = signal-to-noise ratio.

$\langle\dot{\phi}_1\ \dot{\phi}_2\rangle$ = autocorrelation of the deterministic signal.

$O(a_o{}^6)$ = a complex function of the signal-to-noise ratio.

The main point in discussing this relationship is that the deterministic part of the autocovariance function is reduced by a factor of the square of the ordinary power signal-to-noise ratio. Since this solution is only valid for fractional values (i.e., S/N << 1), the deterministic portion of the solution is negligible to a very good degree of approximation for most small values of signal-to-noise ratio.

DISCUSSION OF RESULTS

The solutions that have been presented for the FM portion of this section (and of course also the phase detection portion) represent only partial solutions. Ideally, it would be desired in the future to obtain a more detailed solution for the general amplitude case and certain restricted (i.e., simplified) interfering signals. This, however, appears to be a formidable task.

SECTION 14

GENERALIZATIONS AND EXTENSIONS

INTRODUCTION

SECTIONS 4 to 13 discussed the detailed derivations of the communication analysis step for a number of desired and undesired signal problems. As outlined in Figure 1-1 this constitutes only the first step in the interference evaluation for these problems. The second important step of interference evaluation is that of degradation analysis. It is therefore paramount that the final section of this report discuss the transition of the analysis that has been obtained from Step 1 to that of Step 2. The starting point for a discussion of this transition is a generalization of the approach adopted within this report for communication modeling.

COMMUNICATION GENERALIZATIONS

The analysis of communication problems is inherently difficult due to the different types of signal modulation. It is, however, through this modulation analysis that the primary unknown interference effect to communication systems is obtained. It is, therefore, first desired to obtain the demodulated or detected time amplitude output of all combinations of desired and off-tuned undesired signals

It should be apparent that the time-amplitude output (or its Fourier transform) is desired since this, and only this, function contains all possible information. Therefore, from this and only this can any possible type of problem be solved. This answer can only be obtained, however, for the assumed idealistic case of deterministic signals. For non-deterministic signals, or those in which random fluctuations due to noise or other random parameters have been added, the best that can be obtained is the nth order probability density function. However, in most cases it will probably only be possible to obtain the autocorrelation function or its Fourier transform, the power spectrum. The basic limiting factor in all the analysis problems are the best mathematical outputs that can be obtained. The problem is, however, also practically limited by the variation of the parameters necessary to describe the system. The general procedure is, therefore, to first attempt a complete solution where the output is limited by mathematical complications and then restrict this output by the variation of the known system parameters. If there are few or no parameter limitations the answer that is obtained is the most accurate that can be obtained. This type of analysis is typical of that performed for a particular piece of equipment for a particular project task. When a general ana-

lysis is required and the necessary system parameters are not known a less accurate answer is obtained. It is also apparent that a point is reached before zero information about the equipment is known where no practical interference prediction is possible, due to the wide variance in the output answer. The first output for the communication modeling task consists of a table organizing the detector output for various combinations of desired and undesired signals. This table is shown symbolically in the left-hand portion of Figure 14-1. TABLE 2-2 is an expanded version of this table, an index of the key detection equations found in the appendix of this report.

The second part of the task is to reduce these equations due to the effects of a particular type of system to the point where they can be related to an appropriate intelligibility function. In general it appears that the time-amplitude description of the system output signal contains many complex types of interfering signals and cannot be simply related to any one intelligibility function. However, for the cases of a large signal-to-interference ratio or a large interference-to-signal ratio these functions in most cases reduce to a simpler form. In particular, it is usually possible to directly relate these functions to an appropriate intelligibility function.

Although this section outlines a procedure for coupling the input signals directly to the appropriate intelligibility function, and although this report derives partial intelligibility functions in many cases, it basically does not derive or obtain intelligibility functions. This is considered part of the second task. However, the requirements of communication systems for particular types of subjective intelligibility functions will be extended in the future based largely upon the analysis contained in this report.

The analysis of the communication problem outlined thus far is shown in the complete block diagram of Figure 14-1. For the next part of the operation, it is necessary to obtain the transformation functions relating the average signal-to-interference at the output $[(\overline{S}/\overline{I})_o]$ to the average signal-to-interference at the input $[(\overline{S}/\overline{I})_I]$. This is obtained from the idealized input signal description, the detection table, and appropriate moment calculations. The output at this point consists of a number of transfer functions describing the detection process as a function of the design parameters.

At this point it is also necessary to take into account the offtuning effect of both the IF and the low-pass amplifier. This can generally be taken into account in a similar manner as the

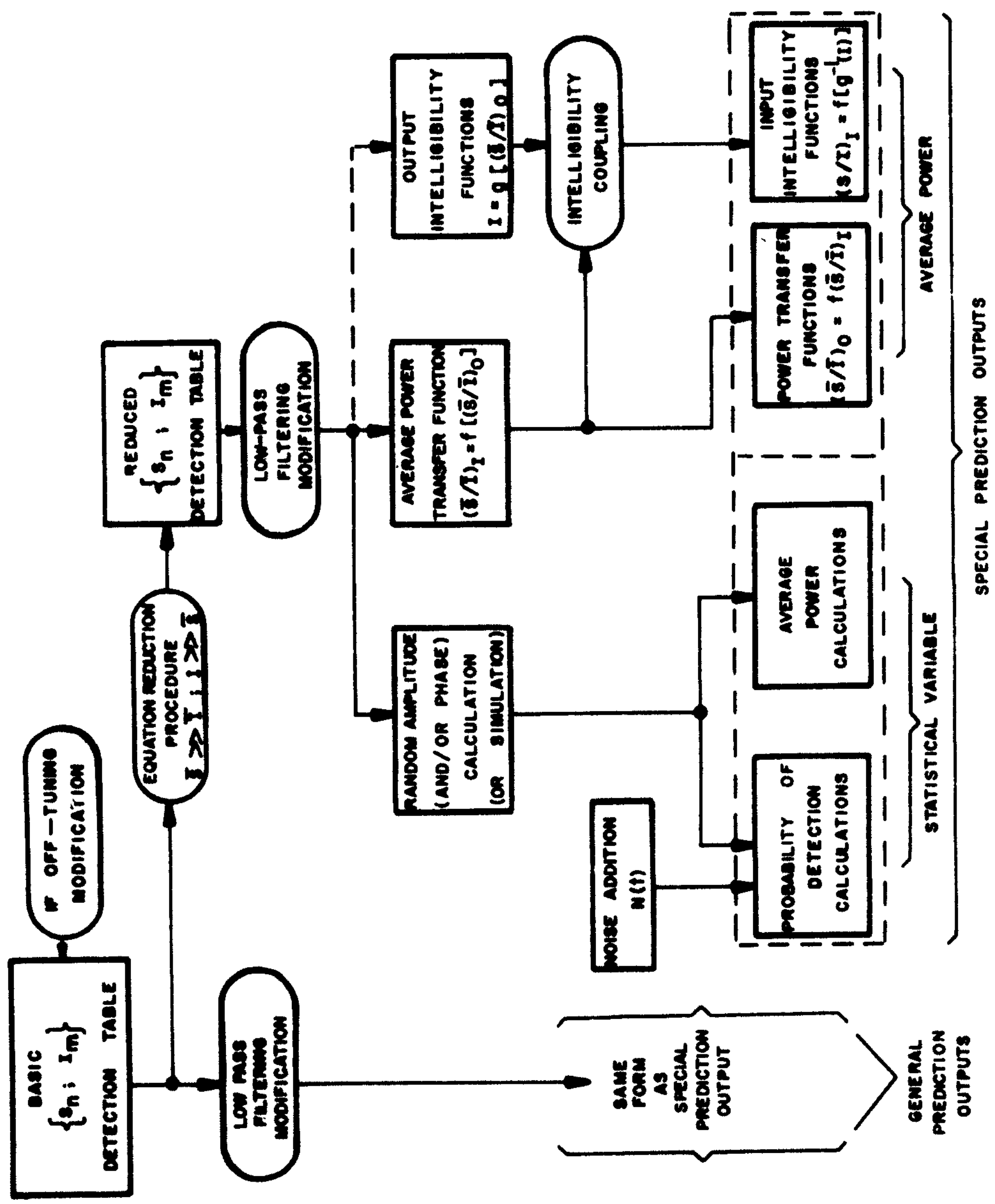


Figure 14-1. Communication System Modeling Outputs

detector. An introductory discussion of this topic was given in reference (7), "Off-Tuning Effects Produced by Interference". A further discussion of this topic is given in APPENDIX III. The basic conclusion drawn from these reports is that the unsymetrical effects of off-tuned interfering signals can be taken into account through a secondary correction factor to the basic detector equations previously discussed. The primary power reduction of the off-tuned signal can be taken into account through an appropriately reduced carrier amplitude obtained from an IF transfer function or selectivity curve.

The output discussed thus far consists of a general n x m detection table and a reduced detection table (obtained from the approximation of large and small signal-to-interference ratios) containing the secondary effects of off-tuning. It also consists of power transfer functions describing each subsection of the basic model. The output at this point is fairly general and can be used for different types of interference prediction purposes. The remainder of this section discusses the extension of these ideas to the evaluation of degradation.

EXTENSIONS OF COMMUNICATION ANALYSIS TO DEGRADATION ANALYSIS

The analysis contained within this report does not directly obtain mathematical models of degradation outputs, but rather, obtains models which are intermediate to the overall problem. Considerations that are drawn from this analysis therefore directly reflect the analysis of these intermediate outputs and attempt to extend the continuation of this task to the analysis of degradation.

The analysis of communications systems were made by studying various one-to-one combinations of desired and undesired or non-design signal types. Brief comments about each combination are given in TABLE 14-1, which lists the signal types studied. Other considerations were obtained by studying the major equations used. The reference numbers for these equations are given in TABLE 2-1, which is an index of the key detection equations used in this study and TABLE 2-2 which is an index of the equations expressing communication outputs.

Only general degradation considerations involving pulsed modulation, analog modulation, and voice modulation, are separately discussed in this section. The section is ended by a discussion of future or advanced degradation modeling concepts.

PULSED MODULATION

Although the analysis of interference to pulsed modulation

TABLE 14-1
SOLUTION TYPES FOR DESIRED AND UNDESIRED SIGNAL COMBINATION

Desired Signal Type	Undesired Signal Type	Solution Types
AM	AM	Solution given both desired and undesired signal tone modulated
FM	AM	Solution for both signals tone modulated
SSB	SSB	Solution for desired and undesired signals with general modulation
Multiplex	AM & FM	Limited to signal-to-interference power calculations
AM	Pulse	Solution with RMS equivalent desired signal modulation
FM	Pulse	Solution with RMS equivalent desired signal modulation
SSB	Pulse	Solution for desired and undesired signals with general modulation
Pulse	Pulse	Probability calculation; computer solution obtained
FSK	Pulse	Probability calculation; computer solution obtained
AM	Noise + Interference Carrier	Solution obtained for a tone modulation desired signal
FM	Noise + Interference Carrier	Solution obtained for large or small carrier-to-noise condition
PM	Noise + Interference Carrier	Solution obtained for large or small carrier-to-noise condition

systems requires further work before the complete degradation solution of a particular system can be obtained, the solutions presently available can be considered to represent nearly complete (i.e., perhaps 90%) solutions. The outputs obtained thus far from this problem consist of the probability of false dismissal (or equivalently the probability of detection) and the probability of false alarm. In themselves, both of these represent nearly complete degradation measures and can often be used for such. However, to carry out a complete degradation evaluation since they are intermediate results, their combined error probabilities are weighed to obtain the average weighted error probabilities and error rates. It is also desirable to consider error rates associated with sentences and messages as a function of the basic bit rates to complete digital modulation performance analysis.

ANALOG MODULATION

For analog modulation systems (such as facsimile) the final measure of the extent of degradation wrought by interference must be made by subjective assessment. In analytical work on this problem the trend has been generally to use the mean square error as a measure of degradation. This has usually been for reasons of mathematical tractability and is only optimum for gaussian noise. However, the mean square error is not an end in itself; it is only an intermediate result useful as an aid in analyzing combinations of desired and undesired signals. When the treatment of the analog cases in this study are made on the premise that desired and undesired signals are independent,* the mean square error problem reduces to that of obtaining the power of the desired and undesired signals. When the solution is reduced to this level, it is apparent that it will yield no more degradation information (other than absolute level) than is obtained by simply determining the power ratio of signal to interference. Therefore, subjective testing of specific criteria (e.g., picture quality of a facsimile transmission) is still required to determine its association with a particular mean square error criteria.

VOICE MODULATION

For voice modulation systems, the same as for analog modulation, the extent of degradation resulting from interference must be subjectively measured. This does not mean that a machine com-

*For many cases there appears to be no justification for analysis to consider anything but independent signals; in others this may constitute the real problem.

putation cannot be used. The basic implication as pointed out before is that in order to verify a machine computation a subjective test must be made. For an analytic study the machine computation could actually be performed mathematically. However, a subjective test must still be performed to verify this process. It is in this context that the remaining discussion refers only to subjective tests.

At present the only way to gain an accurate notion of degradation to any particular system (i.e., the percent of words correctly received as a function of desired and undesired signals) is to subject it to a test. Allowing a separate test for each system, the number of tests required would be voluminous. However, this study shows that this straightforward approach is not necessary and that through analytical techniques the number of tests needed can be reduced to a more reasonable level. Although this report does not include a complete study of all interference cases, it is apparent that the undesired signal outputs of voice systems will contain certain mathematical similarities, regardless of signal types involved. In particular, the predominant interfering output signal for all modulation types is modulated beat tones. The amplitude and spectra associated with these beat tones are functions of the type of modulation (AM, FM, etc.) and the parameter of modulation (which is generally the modulation index).

Although the problem of specifying an exact degradation test in terms of these tones is not easy, due to the numerous beat tones and their complicated sideband structure, considerable simplification can be obtained because the same type of phenomena results for all types of modulated signals. Therefore, the same type of degradation testing can be used for AM, FM, SSB, and phase modulated signals.

Another major conclusion drawn from the analysis of voice as well as analog modulation is that it is necessary to consider the effect of the audio filter when calculating interference effects. Without this narrowband filter action, all the beat terms in the interference output signal would have to be considered. In practice, only a few, sometimes only one, are usually significant and therefore need be considered.

With respect to off-tuned interference, the meaning of "significant" should be elaborated and is arrived at as follows: If a constant input signal-to-interference ratio is maintained and the off-tuning of the interfering signal is varied, an intelligibility function similar to that shown in Figure 14-2 will be obtained. This curve is a general symbolic representation and does not repre-

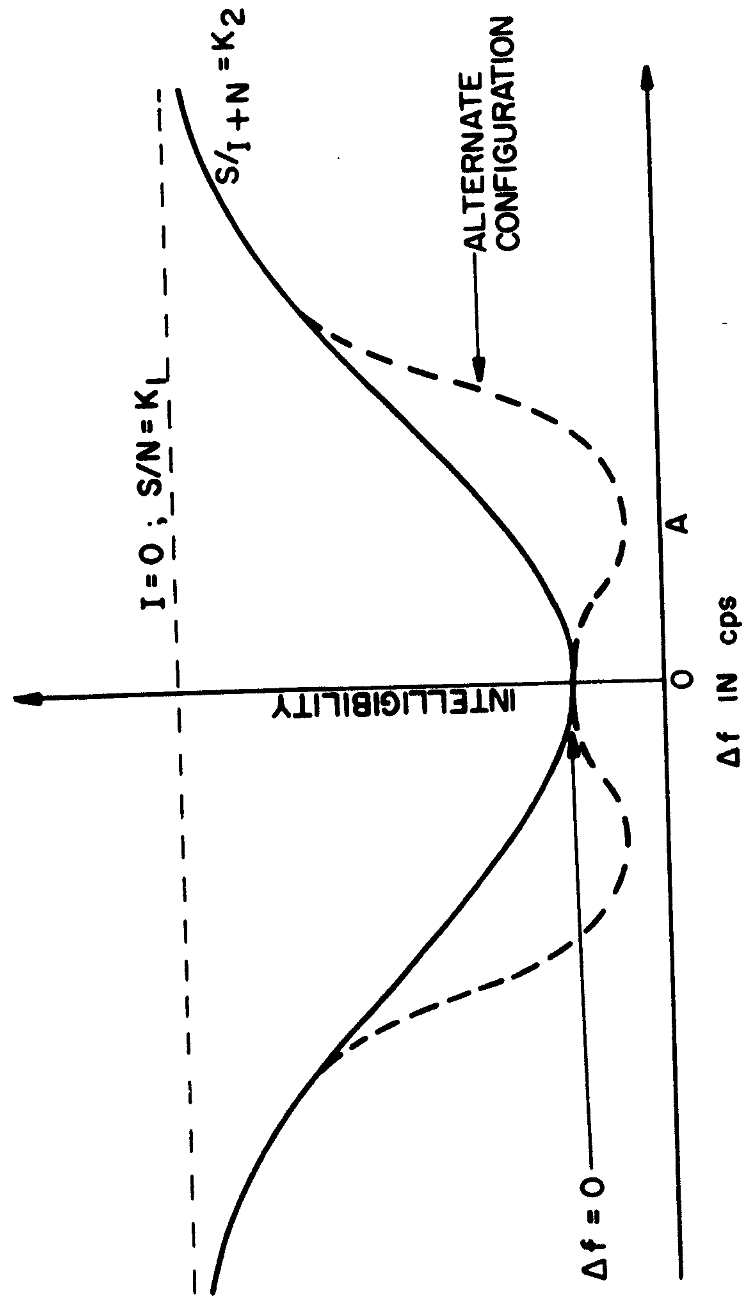


Figure 14-2. Off-Tuned Intelligibility Function

sent any particular modulation type. It should be noted that there is a possibility of decreasing intelligibility as a function of the off-tuning although the general trend is to increase the value of the intelligibility function.

Considering that the general trend depicted does exist, it is appropriate to make certain assumptions concerning the off-tuned detected signals. The first case of interest is that in which the off-tuning is zero. Then, for all modulation cases, the output signal is produced by the sideband terms of the desired and undesired signal. Therefore, for all modulation cases, all that is required for degradation evaluation is to test low-pass interference modulation (possibly of an uncorrelated speech type) where the amplitude has been obtained from the appropriate modulation equations of this report. When the general off-tuned case is considered, this test generalizes to testing AM and FM signals centered at the off-tuned frequencies. For these cases the modulated sidebands appear in general to be less significant due to the low-pass filter action. Retaining their exact characteristics therefore does not appear to affect the first order signal calculations, providing the correct total power is used. The ordinary modulation index to be used for this testing is also determined by using the proper interference detection equations from this report.

ADVANCED DEGRADATION MODELING

For future or advanced degradation modeling, a brief study was made of the equations derived within this report. Since not all cases have been covered, and those covered have not always been analyzed to a desired completion, absolute or complete answers are not available, but this study leads to the following conclusions:

1. The IF-filtered interference carrier power is the key interfering system performance parameter for most systems. The reasons for making this conclusion were brought out in SECTIONS 4 through 13 of this report. The importance of this conclusion is in what it implies: that only one, and not a large number, of power calculations need to be performed at IF in order to predict co-channel degradation. (Even for a system without a carrier such as SSB, a single tone or pseudo-carrier model is usually used.)

2. The difference in frequency between the desired and undesired signal (Δf) is a key parameter in determining the amount of filtered low-pass output power (a term monotonically related to degradation). This is also true at IF but is even more critical at the low-pass filter, due to the nonlinearity of the detector and the usual narrowness of the low-pass filter relative to the

IF. The specification of the off-tuning is therefore required for a specific performance analysis.

3. The constant phase difference between the desired and undesired signals did not affect the final system output equations for all cases considered within this report. This is basically due to the desired system output, since for the systems considered the random phase variation has been statistically eliminated.*

4. The degree to which the filter characteristics are known greatly affects the ability to predict total system performance. For all except digital signals, when the probability of false dismissal and false alarm constitute an adequate performance measure, the filter characteristics of the IF and the low-pass filter as well as the ideal detector type are required to make an accurate performance evaluation.

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*This is not generally true as in the case of an FM or phase synchronous system.

APPENDIX I

DETECTION MODELING OF DETERMINISTICALLY DESCRIBED SIGNALS

INTRODUCTION

For five major types of detectors, this appendix discusses models with which the effects of combinations of desired and undesired signals can be predicted, if deterministic descriptions are available for the input signals to the receiver. The basis of each model is a mathematical expression of the time-amplitude output of the detector.

This appendix omits consideration of all circuits prior to the detector and begins by considering the signal as it appears at the input to the detector. This appendix ignores the direct effects of thermal noise, by assuming that all signals at the receiver input have a large carrier-to-noise ratio (C/N)* so that the first order effects of noise can be neglected or taken into account separately. Appendix II presents the derivation of the output noise power for the various types of detectors and the large carrier-to-noise condition. Consequently the combination of APPENDIXES I and II can be used to obtain the first order signal-to-interference-plus-noise ratios. The applicability of the models is limited by only a small region of signal-to-noise ratio in which the analysis does not apply. When the general noise case is not considered it also results in the advantage of holding the complexity of the analysis to a feasible level. In fact, taking noise into account would in many cases increase the complexity of the problem beyond the point of practical analysis.**

GENERAL DETECTOR OUTPUT CONSIDERATIONS

The input signals to the second detector consist of a narrowband waveform. A narrowband waveform can be described by the complex form.

$$v(t) = \mathrm{R1}\left[\left(V(t)\, e^{j\phi_v(t)}\right) e^{j(\omega_0 t + \phi_0)}\right] \qquad \text{(I-1)}$$

*A condition approximately met when C/N ≥ 6 db.

** A discussion of the general approach and its difficulty is given in SECTION 13.

where,

Rl means take the real part of.

$V(t)$ = a slowly varying function compared to ω_o.

ω_o = radian carrier frequency.

$\phi_V(t)$ = slowly varying phase function compared to ω_o.

ϕ_o = carrier phase.

In the present problem, combinations of narrowband signals through a nonlinear device will be considered. Superposition, therefore, does not apply and a single equivalent signal must be obtained. It will consequently be more convenient to rewrite equation (I-1) in the equivalent quadrature forms

$$v(t) = \left[V(t) \cos (\phi_V(t)+\phi_o)\right] \cos \omega_o t - \left[V(t) \sin (\phi_V(t)+\phi_o)\right] \sin \omega_o t \tag{I-2a}$$

$$= X(t) \cos \omega_o t - Y(t) \sin \omega_o t \tag{I-2b}$$

which can also be written (see Figure I-1)

$$v(t) = \left[X^2(t) + Y^2(t)\right]^{\frac{1}{2}} \cos \left[\omega_o t + \phi(t)\right] \tag{I-3a}$$

$$v(t) = |V(t)| \cos \left[\omega_o t + \phi(t)\right] \tag{I-3b}$$

where

$$\phi(t) = \tan^{-1} \frac{Y(t)}{X(t)} \tag{I-3c}$$

and

$|V(t)|$ is used since only the positive square root is considered.

The types of detectors to be considered consist of linear envelope(where envelope is defined as the slowly varying portion of $V(t)$, i.e. $|V(t)|$), square law envelope, phase modulation, frequency modulation and synchronous detectors. The zero-zone output or the unfiltered low-pass output of these detectors can ideally be described by the following:

1. Half-wave linear envelope detector.

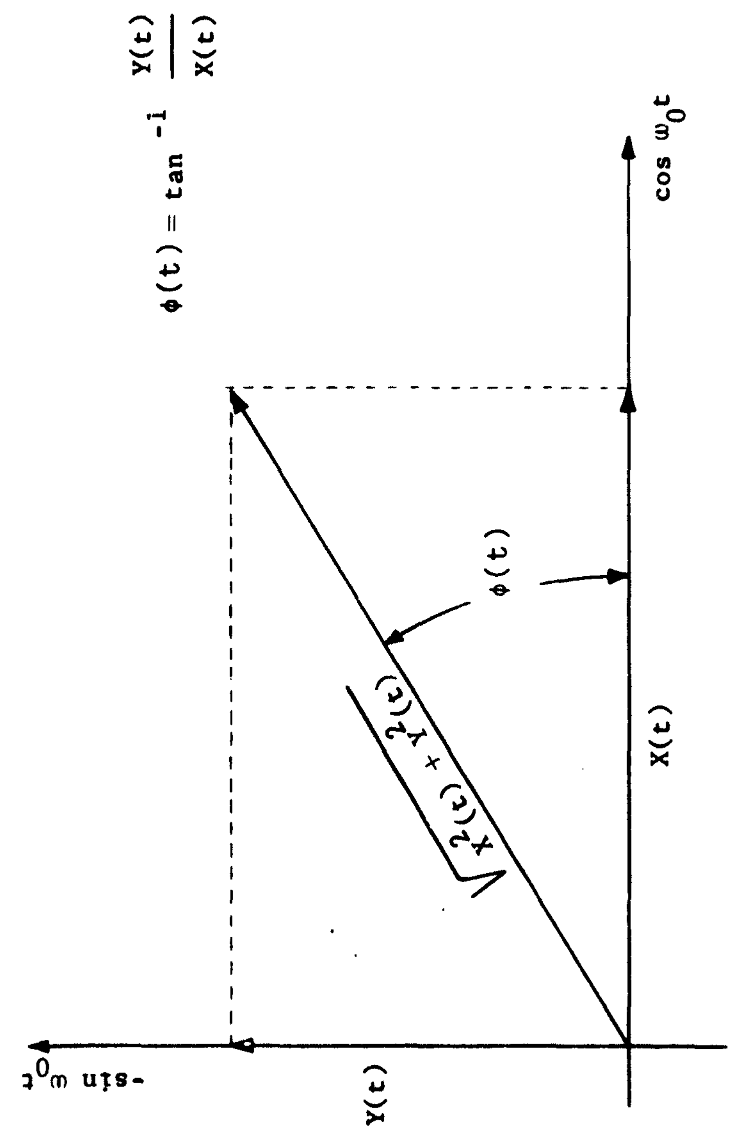

Figure I-1. Vector Signal Relationships.

$$v_o(t) = K_{LIN}\left[|V(t)| - \overline{|V(t)|}\right] \tag{I-4}$$

where,

K_{LIN} = linear detector constant.

$\overline{|v(t)|}$ = the average linear output (i.e. the DC term)

2. Half-wave square law envelope detector.

$$v_o(t) = K_{SQ}\left[|V(t)|^2 - \overline{|V(t)|^2}\right] \tag{I-5}$$

where,

K_{SQ} = square law detector constant.

$\overline{|v(t)|^2}$ = the average square output (i.e. the average power)

3. Phase detector.

$$v_o(t) = K_\phi . \phi(t) \tag{I-6}$$

where,

K_ϕ = phase detector constant.

4. Frequency detector*

$$v_o(t) = K_{FM}\left[\frac{1}{2\pi} \cdot \frac{d\phi(t)}{dt}\right] \tag{I-7}$$

where,

K_{FM} = frequency detector constant.

5. Synchronous detector.

$$v_o(t) = K_{SYN}\left[|V(t)| \cos\left(\omega_o t + \phi_R\right)\right] \tag{I-8}$$

where,

K_{SYN} = synchronous detector constant.

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*In the special cases in which it is desired to detect the carrier, f_o is added to this formulation.

and,

$$\phi_R = \text{synchronous carrier phase.}$$

The description of these ideal detectors can be found in numerous references and the reader is referred to references (5), (8), and (9) for a discussion of their derivation. This topic is also discussed in Appendix B of reference (11). The output signals described in equations (I-4) through (I-8) are the unfiltered low-pass outputs. If the output signal is passed through a low-pass output filter, the output is given by

$$v_o'(t) = \int_{-\infty}^{\infty} h(\tau)\, v_o(t-\tau)\, d\tau \qquad \text{(I-9)}$$

where

$h(\tau)$ = impulse response of the low-pass zonal filter.

The purpose of this appendix is not to present an investigation of filtered output responses. This topic is discussed within the individual sections of this report. Also, it is more useful to leave the constant terms in equations (I-4) and (I-5). Then the final derived signal is expressed as $|V(t)|$ or $|V(t)|^2$ in the cases of the linear and square law detectors. It will still be necessary to carry out the operation of equation (I-9) on the output signal, $v_o(t)$, to obtain the output, $v_o'(t)$, when the filtered output is desired. These calculations are carried out within those sections of the report in which particular systems and consequently particular filters are considered.

The input signal to the second detector for interfering situations consists of a desired signal, S(t), and an interfering signal, I(t). The additive input is given by

$$v(t) = S(t) + I(t) \qquad \text{(I-10a)}$$

where, from the previous discussion,

$$S(t) = X_S(t) \cos \omega_o t - Y_S(t) \sin \omega_o t \qquad \text{(I-10b)}$$

$$I(t) = X_I(t) \cos \omega_o t - Y_I(t) \sin \omega_o t \qquad \text{(I-10c)}$$

The combined signal is therefore

$$v(t) = \left[X_S(t) + X_I(t)\right] \cos \omega_o t - \left[Y_S(t)+Y_I(t)\right] \sin \omega_o t \qquad \text{(I-11a)}$$

$$= X(t)\cos\omega_o t - Y(t)\sin\omega_o t \qquad \text{(I-11b)}$$

$$= \left[X^2(t)+Y^2(t)\right]^{\frac{1}{2}}\cos\left[\omega_o t+\phi(t)\right] \qquad \text{(I-11c)}$$

It is the purpose of this appendix to obtain the slowly varying amplitude and phase portions of equation (I-11) for various combinations of desired signals and interference and to operate on these resultant signals by the previously described detector functions.

The analysis has thus far not considered the effect of thermal, atmospheric, or receiver noise on the desired signal, since a large carrier-to-noise condition was hypothesized. It is, however, sometimes desired to calculate the output signal-to-noise ratio so that the output noise power must be obtained from the input noise to a particular detector. The condition of independence between signal and noise exists for this situation. This problem will be discussed in APPENDIX II.

This appendix presents expressions for the unfiltered low-pass zonal output of the previously considered detectors for various combinations of interfering signals. The detector outputs for various combinations of deterministic desired and undesired signals is discussed next.

SQUARE LAW DETECTION

The first problem to be considered is that of the square law detector. The analysis of this detector is useful since:

1. A complete solution can be obtained.
2. The square law is an ideal mixer.
3. The solution of the linear detector can be obtained from the square-law detected output.
4. This model can be used for all detector types when the noise is much greater than the signal.

In order to compute the output of a square law detector it is desired to obtain

$$|V(t)|^2 = X^2(t) + Y^2(t) \qquad \text{(I-12)}$$

where the interference and the desired signal are to be written in the form

$$S(t) = X_S(t)\cos\omega_o t - Y_S(t)\sin\omega_o t \qquad \text{(I-13)}$$

$$I(t) = X_I(t)\cos\omega_o t - Y_I(t)\sin\omega_o t \qquad \text{(I-14)}$$

The deterministic waveform for an AM waveform is

$$S_{AM}(t) = A_S(1+\sum_{K=1}^{N} m_{SK} \cos \omega_{SK}t) \cos \omega_o t \qquad (I-15)$$

where

A_S = peak carrier amplitude.

m_{SK} = amplitude of the Kth harmonic divided by A_S.

ω_{SK} = the Kth intelligence harmonic.

<u>Square Law Detector, Application 1, A Desired Tone-Modulated AM Signal and an Undesired CW Signal.</u> The desired signal will be treated as a single pair of sidebands given by

$$S_{AM}(t) = A_S(1+m_S \cos \omega_S t) \cos \omega_o t \qquad (I-16a)$$

$$= A_S \cos \omega_o t+\frac{A_S m_S}{2} \cos (\omega_o-\omega_S)t+\frac{A_S m_S}{2} \cos (\omega_o + \omega_S)t. \qquad (I-16b)$$

The interfering signal will be assumed to be of the CW form

$$I_{CW}(t) = A_I \cos\left[(\omega_I t) + \theta_I\right] \qquad (I-17a)$$

$$= A_I \cos\left[(\omega_o+\Delta\omega)\ t + \theta_I\right] \qquad (I-17b)$$

where,

$\omega_I = \omega_o + \Delta\omega$= interference carrier frequency.

$\Delta\omega$= off-tuned frequency.

θ_I= the interfering carrier phase relative to the desired signal.

The $\Delta\omega t + \theta_I$ terms are the troublesome terms encountered in all interference problems. These terms change the even function in terms of (ω_I) to an even and an odd function in terms of ω_o.

It is convenient to rewrite equation (I-17) in the quadrature form

$$I_{CW}(t) = \left[A_I \cos (\Delta\omega t + \theta_1)\right] \cos \omega_o t - \left[A_I \sin (\Delta\omega t + \theta_I)\right] \sin \omega_o t \qquad (I-18)$$

Combining signal and interference

$$v(t) = S_{AM}(t) + I_{CW}(t) \tag{I-19a}$$

$$= \left[A_S(1 + m_S \cos \omega_S t) + A_I \cos (\Delta\omega t + \theta_I)\right] \cos \omega_o t$$

$$- \left[A_I \sin (\Delta\omega t + \theta_I)\right] \sin \omega_o t \tag{I-19b}$$

The ideal unfiltered square law detected output is

$$|V(t)|^2 = A_I^2 + A_S^2(1 + m_S \cos \omega_S t)^2$$

$$+ 2A_S A_I(1 + m_S \cos \omega_S t) \cos(\Delta\omega t + \theta_I) \tag{I-20a}$$

$$= \left[A_I^2 + A_S^2 + \frac{m_S^2 A_S^2}{2}\right] + 2 A_S^2 m_S \cos \omega_S t$$

$$+ \frac{m_S^2 A_S^2}{2} \cos 2 \omega_S t$$

$$+ 2 A_S A_I \cos (\Delta\omega t + \theta_I)$$

$$+ m_S A_I A_S \cos \left[(\omega_S + \Delta\omega)t + \theta_I\right]$$

$$+ m_S A_I A_S \cos\left[(\Delta\omega - \omega_S)t + \theta_I\right] . \tag{I-20b}$$

This output signal can be presented in a number of ways. The spectrum could be graphed. The desired output signal could be plotted as a function of the undesired carrier to the average desired signal, or plotted as the ratio of total undesired to the average desired signal. Whatever are the parameters of the output, it is intended to show the interference effects of the undesired signals and is basically a system performance problem which is discussed within the appropiate section of the report.

The form of equation (I-20) indicates that both the amplitude and frequency of the detected interference play key roles in interference degradation, and their roles are difficult to separate. Since it is not the purpose of this discussion to evaluate interference effects, equation (I-20b) expresses the output in the form most suited to the general nature of this appendix. The dc terms

are, of course, usually removed by the low-pass filter.

It is sometimes more convenient to normalize equation (I-20b) in the form

$$\begin{aligned}\frac{|V(t)|^2}{A_S^{\,2}} = 1 &+ \frac{m_S^{\,2}}{2} + R_I^{\,2} + 2\, m_S \cos \omega_S t \\ &+ \frac{m_S^{\,2}}{2} \cos 2\,\omega_S t \\ &+ 2R_I \cos(\Delta\omega t + \theta_I) \\ &+ m_S\, R_I \cos\left[(\omega_S t + \Delta\omega)t + \theta_I\right] \\ &+ m_S\, R_I \cos\left[(\Delta\omega - \omega_S)t - \theta_I\right]\end{aligned} \qquad \text{(I-21)}$$

where

$$R_I = A_I/A_S$$

For the typical case when $m_S = .3$ and $R_I \ll 1$, this can approximately be reduced to

$$\frac{|V(t)|^2}{A_S^{\,2}} \approx 1 + .6 \cos \omega_S t + 2\, R_I \cos(\Delta\omega t + \theta_I) \qquad \text{(I-22)}$$

This, in turn, exemplifies the important nature of the beat tone in interference. If θ_I is treated as a random variable, and particularly in the case in which all values of phase are assumed equally likely, the filtered output interference power is only a function of R_I and $\Delta\omega$.

The derivation of equation (I-22) is limited by the condition that $R_I \ll 1$. This approximate answer is a very good representation of the output signal when $R_I \leq .32$ (-10db) and still representative when $R_I = .5$ (-6db). When $A_I \gg A_S$ and consequently $R_S = A_S/A_I \ll 1$ the same reasoning applies and therefore a region from approximately -6 db to +6 db in the signal-to-interference space is undefined or not representable by the form of equation (I-22). The interconnecting function between these two end points could further be approximated by a straight line so that the actual restriction implied by the large carrier condition is not as

restrictive as it might appear.

Square Law Detector, Application 2, A Desired CW Signal and an Undesired Pulsed Signal. Let the undesired signal consist of a pulsed CW. The signal can therefore in general be written as

$$I(t) = A_I(t) \cos (\omega_o t + \Delta\omega t + \theta_I). \qquad \text{(I-23)}$$

Consider the simplest case of a desired CW signal,

$$S(t) = A_S \cos \omega_o t \qquad \text{(I-24)}$$

The combined signal is given by

$$V(t) = S(t) + I(t) \qquad \text{(I-25a)}$$

$$= \left[A_I(t) \cos (\Delta\omega t + \theta_I) + A_S\right] \cos \omega_o t$$
$$- \left[A_I(t) \sin (\Delta\omega t + \theta_I)\right] \sin \omega_o t \qquad \text{(I-25b)}$$

The square law detected output is found to be

$$\frac{|V(t)|}{A_S^2} = 1 + R_I^2(t) + 2R_I(t) \cos (\Delta\omega t + \theta_I) \qquad \text{(I-26a)}$$

$$= 1 + R_I^2 A_{IN}^2(t) + 2R_I A_{IN}(t) \cos (\Delta\omega t + \theta_I) \qquad \text{(I-26b)}$$

where

$$A_{IN}(t) = \frac{A_I(t)}{A_I} = \text{the normalized pulse signal}$$

Square Law Detector, Application 3, A Desired AM Tone-Modulated Signal and an Undesired Pulsed Signal. For the case of pulsed CW interference to an AM tone modulated system the signals can be written

$$S(t) = A_S(1 + m_S \cos \omega_S t) \cos \omega_o t \qquad \text{(I-27)}$$

$$I(t) = A_I A_{IN}(t) \cos \left[(\omega_o + \Delta\omega) t + \theta_I\right] \qquad \text{(I-28)}$$

The combined signal becomes

$$V(t) = \left[A_S^2 (1 + m_S \cos \omega_S t)^2 + 2A_S(1 + m_S \cos \omega_S t) A_I A_{IN}(t) \cos (\Delta\omega t + \theta_I)\right]^{\frac{1}{2}} \cos (\omega_o t + \phi) \qquad \text{(I-29)}$$

The normalized square law detected output is found to be

$$\frac{|V(t)|}{{A_S}^2} = 1 + \frac{{m_S}^2}{2} + 2\, m_S \cos \omega_S t + \frac{{m_S}^2}{2} \cos 2\, \omega_S t \qquad \text{(I-30)}$$

$$+ 2\, R_I\, A_{IN}(t) \cos (\Delta\omega t + \theta_t)$$

$$+ m_S R_I A_{IN}(t) \cos \left[(\Delta\omega \pm \omega_S)t + \theta_t\right]$$

Square Law Detector, Application 4, A Desired and Undesired AM Signal. As the next example consider the general AM modulated desired and undesired signals

$$S_{AM}(t) = A_S(1 + \sum_{K=1}^{N} m_{SK} \cos \omega_{SK} t) \cos \omega_o t \qquad \text{(I-31a)}$$

$$= A_S\left[1 + S_K(t)\right] \cos \omega_o t \qquad \text{(I-31b)}$$

$$I_{AM}(t) = A_I\left[1 + \sum_{K=1}^{N} m_{IK} \cos \omega_{IK} t\right] \cos \left[(\omega_o + \Delta\omega)t + \theta_I\right] \qquad \text{(I-32a)}$$

$$= A_I \left[1 + I_K(t)\right] \cos \left[(\omega_o + \Delta\omega)t + \theta_I\right] \qquad \text{(I-32b)}$$

The modulation coefficients m_{SK} and m_{IK} are here considered as Fourier coefficients, since it is convenient to represent the signal as a periodic function. Another form of this equation could be written in which an average modulation factor was considered. If the modulation coefficient is considered as a constant (perhaps obtained by computing the root mean square signal amplitude) for more than a few pairs of sidebands, little physical interpretation is seen. The signal for a constant coefficient is given by

$$f_S(t) = \overline{m_S} \sum_{K=1}^{N} \cos \omega_K t \qquad \text{(I-33)}$$

This is equivalent to picking a periodic impulse train (in the limit as $N \to \infty$) as the signal representation. It therefore appears that the general Fourier representation is necessary for most complex signals.

From equations (I-31b) and (I-32b), the general combined signal is found to be

$$v(t) = S_{AM}(t) + I_{AM}(t) \tag{I-34a}$$

$$= \{A_S\left[1 + S_K(t)\right] + A_I\left[1 + I_K(t)\right]\cos(\Delta\omega t + \theta_I)\}\cos\omega_o t$$
$$- \{A_I\left[1 + I_K(t)\right]\sin(\Delta\omega t + \theta_I)\}\sin\omega_o t \tag{I-34b}$$

$$= X'(t)\cos\omega_o t - Y'(t)\sin\omega_o t \tag{I-34c}$$

The ideal square law detected output is

$$\frac{|V(t)|^2}{A_S^{\,2}} = \left[1 + S_K(t)\right]^2 + 2R_I\left[1 + S_K(t)\right]\left[1 + I_K(t)\right]\cos(\Delta\omega t + \theta_I) + R_I^{\,2}\left[1 + I_K(t)\right]^2 \tag{I-35a}$$

$$= 1 + R_I^{\,2} + 2\left[S_K(t) + R_I^{\,2} I_K(t)\right] + S_K^{\,2}(t) + R_I\, I_K^{\,2}(t) + 2R_I\cos(\Delta\omega t + \theta_I)\left[1 + S_K(t) + I_K(t) + S_K(t)\, I_K(t)\right] \tag{I-35b}$$

It is apparent from this general expression that attempts to predict interference degradation from the signal-to-total-average-interference ratio can lead to large errors due to the square and $\cos(\Delta\omega t + \theta_I)$ terms. The output is seen to consist of linear terms, square terms, and linear and square terms multiplied by the factor $A_S A_I \cos(\Delta\omega t + \theta_I)$. The general output is also seen to be heavily dependent upon the ratio of the carriers. Two cases of this ratio are actually of interest. These are the cases when

$$A_S >> A_I$$

and

$$A_I >> A_S$$

In the first case, it is convenient to rewrite equation (I-35b) in the form

$$\frac{|V(t)|^2}{A_S^{\,2}} = 1 + R_I^{\,2} + 2\left[S_K(t) + R_I^{\,2} I_K(t)\right] + S_K^{\,2}(t) + R_I^{\,2} I_K^{\,2}(t) + 2R_I\cos(\Delta\omega t + \theta_I)\left[1 + S_K(t) + I_K(t) + S_K(t)\, I_K(t)\right] \tag{I-36}$$

where

$$R_I = A_I/A_S$$

If $R_I << 1$, it is appropriate to write, for a first order magnitude,

$$\frac{|V(t)|^2}{{A_S}^2} = 1 + 2\ S_K(t) + S_K^2(t) + 2\ R_I \cos(\Delta\omega t + \theta_I)\left[1 + S_K(t) + I_K(t) + S_K(t)\ I_K(t)\right] \quad \text{(I-37)}$$

The interfering terms are therefore first order dependent on $R_I \cos(\Delta\omega t + \theta_I)$ and oscillating at a $\Delta\omega$ rate. If $\Delta\omega$ is outside the desired signal passband but not far enough that the linear and cross product term are negligible, the case shown in Figure I-2b will result. The case shown is an example where $S_K(\omega)$ has an idealized square spectrum. If $\Delta\omega = \omega_{S0}$, the case shown in Figure I-2c results. The resultant total loss of performance is highly dependent on $\Delta\omega$ falling within the desired signal band.

In the second case, in which $A_I >> A_S$ we obtain

$$\begin{aligned}\frac{|V(t)|^2}{A_I^2} \doteq\ & 1 + {R_S}^2 + 2\left[R_S^2\ S_K(t) + I_K(t)\right] \\ & + {R_S}^2{S_K}^2(t) + {I_K}^2(t) \\ & + 2\ R_S \cos(\Delta\omega t + \theta_I)\left[1 + S_K(t) + I_K(t) \right. \\ & \left. + S_K(t)\ I_K(t)\right]\end{aligned} \quad \text{(I-38)}$$

where

$$R_S = A_S/A_I$$

Since it is desired to retain the first order desired terms, this can be simplified to

$$\begin{aligned}\frac{|V(t)|^2}{{A_I}^2} =\ & 1 + R_S^2 + 2\ R_S^2\ S_K(t) + 2I_K(t) + I_K^2 \\ & + {R_S}^2{S_K}^2(t)\end{aligned} \quad \text{(I-39)}$$

when

$$R_S << 1.$$

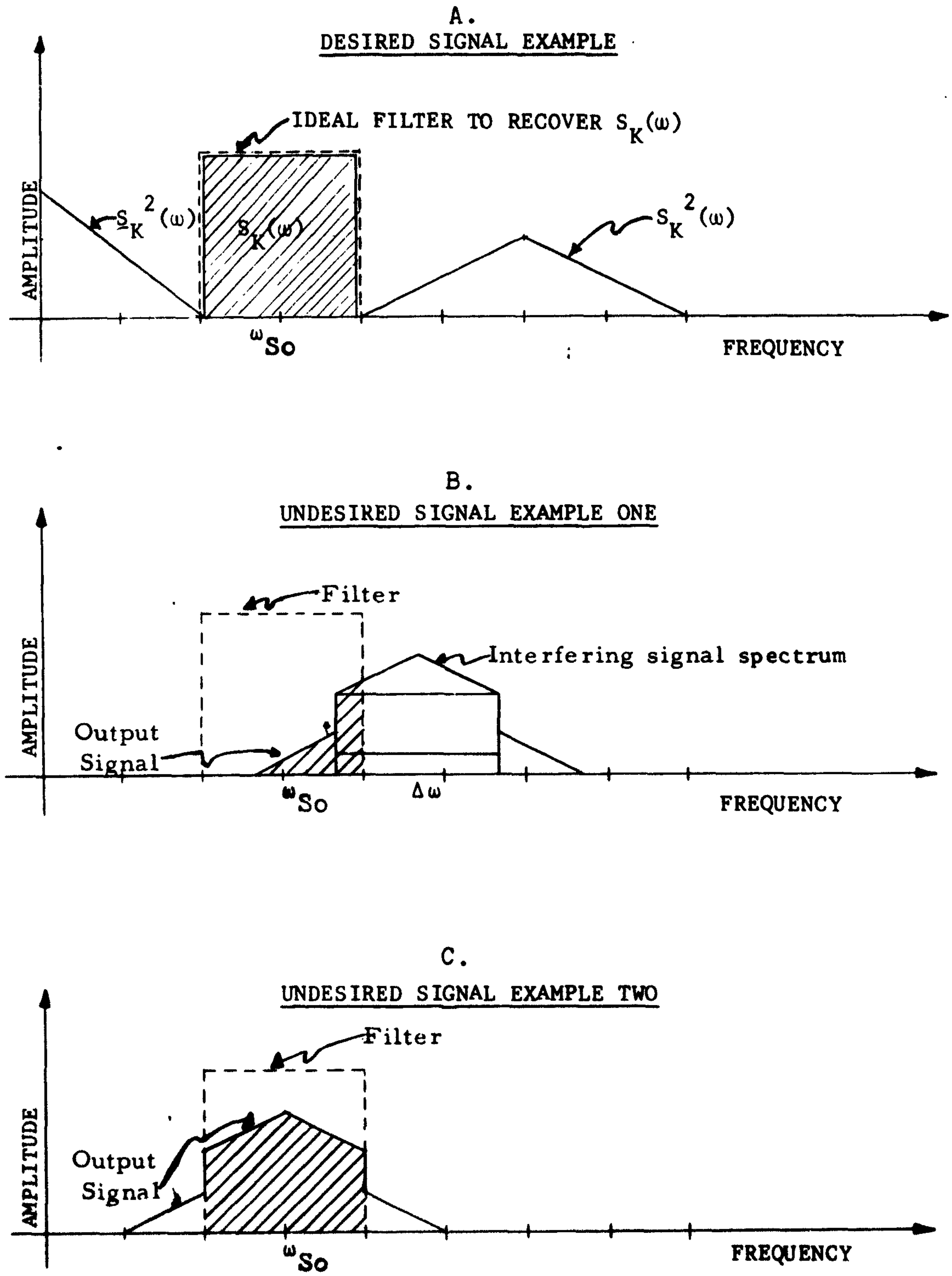

Figure I-2. Off-Tuning Degradation Effects.

The difference between this small signal case and the previously discussed large signal case is the absence of the beat signal.

<u>Square Law Detector, Application 5, a Desired AM Signal with Multiple CW Interfering Signals.</u> The multiple AM interference case could be handled in a straightforward but complicated, expansion of the previous problem. It is apparent that the answer would result in a large number of second order terms involving $I_K(t)$ and first order terms involving $\cos(\Delta\omega t + \theta_I)$ for the case of predominant interest $A_S \gg A_I$. The $I_K(t)$ will, therefore, be neglected and the case of a number of interfering CW wave-forms considered. Let

$$I(t) = \sum_{J=1}^{N} I_J(t) = A_{I1}\cos\left[(\Delta\omega_J + \omega_o)t + \theta_{IJ}\right]$$

$$+ \ldots\ldots A_{IN}\cos\left[(\Delta\omega_N + \omega_o)\,t + \theta_{IN}\right] \qquad \text{(I-40)}$$

The combined desired signal plus interference is

$$v(t) = \sum_{J=1}^{N} I_J(t) + A_S\left[1 + S_K(t)\right]\cos\omega_o t \qquad \text{(I-41)}$$

$$= \{A_{I1}\cos(\Delta\omega_1 t + \theta_{I1}) + \ldots . A_{IN}\cos(\Delta\omega_N t + \theta_{IN})$$

$$+ A_S\left[1 + S_K(t)\right]\}\cos\omega_o t$$

$$- \{A_{I1}\sin(\Delta\omega_1 t + \theta_{I1}) + \ldots . A_{IN}\sin(\Delta\omega_N t$$

$$+ \theta_{IN})\}\sin\omega_o t \qquad \text{(I-42)}$$

The square law detected output is therefore

$$|V(t)|^2 = A_{I1}{}^2 + A_{I2}{}^2 + \ldots . A_{IN}{}^2 + A_S{}^2 + 2\,A_S{}^2 S_K(t)\;A_S{}^2 S_K{}^2(t)$$

$$+ 2\,A_S\left[A_{I1}\cos(\Delta\omega_1 t + \theta_1) + \ldots . A_{IN}(\cos\Delta\omega_N t + \theta_N)\right]$$

$$+ 2\,A_{I1}A_{I2}\cos\left[(\Delta\omega_1 + \Delta\omega_2)t + \theta_1 + \theta_2\right]$$

$$\vdots$$

$$+ 2\,A_{IJ}A_{IN}\cos\left[(\Delta\omega_J + \Delta\omega_N)t + \theta_J + \theta_N\right]$$

$$\vdots$$

$$+ 2\,A_{IN-1}A_{IN}\cos\left[(\Delta\omega_{N-1} + \Delta\omega_N)\,t + \theta_{N-1} + \theta_N\right]$$

$$+ 2\,A_S S_K(t)\left[A_{I1}\cos(\Delta\omega_1 t + \theta_I) + \ldots . A_{IN}\cos(\Delta\omega_N t + \theta_N)\right] \qquad \text{(I-43)}$$

The answer consists of DC terms, the desired and undesired terms, interference cross product terms, and the interference and the desired signal cross product terms. Due to the involved nature of the multiple interference terms, no simplification will be made. Consider, instead, the simpler case of two CW interfering signals.

$$I_{I2}(t) = A_{I1} \cos\left[(\Delta\omega_1 + \omega_o)t + \theta_{I1}\right] + A_{I2} \cos\left[(\Delta\omega_2 + \omega_o)t + \theta_{I2}\right] \tag{I-44}$$

The square law output is therefore readily obtained from equation (I-42) as

$$\begin{aligned}|V(t)|^2 = {} & A_I^2 + A_{I2}^2 + A_S^2 \\ & + A_S^2\left[2\, S_K(t) + S_K^2(t)\right] \\ & + 2\, A_S\left[A_{I1} \cos(\Delta\omega_1 t + \theta_1) + A_{I2} \cos(\Delta\omega_2 t + \theta_2)\right] \\ & + 2\, A_{I1}A_{I2} \cos\left[(\Delta\omega_1 + \Delta\omega_2)t + \theta_1 + \theta_2\right] \\ & + 2\, A_S S_K(t) \left[A_{I1} \cos(\Delta\omega_1 t + \theta_1) \right. \\ & \left. + A_{I2} \cos(\Delta\omega_2 t + \theta_2)\right]\end{aligned} \tag{I-45}$$

Perhaps the most common case is when

$$A_S \gg A_{I1}$$

$$A_S \gg A_{I2}$$

$$A_{I1} \neq A_{I2}$$

For this case equation (I-45) can be approximated and normalized to

$$\frac{|V(t)|^2}{A_S^2} = 1 + 2\, S_K(t) + S_K^2(t)$$ (This equation continued on next page.)

$$+ 2 R_{I1} \cos (\Delta\omega_1 t + \theta_1)$$
$$+ 2 R_{I2} \cos (\Delta\omega_2 t + \theta_2)$$
$$+ 2 S_K(t) \left[R_{I1} \cos (\Delta\omega_1 t + \theta_1)\right.$$
$$\left. + R_{I2} \cos (\Delta\omega_2 t + \theta_2)\right] \qquad \text{(I-46)}$$

where,

$$R_{I1} = \frac{A_{I1}}{A_S},\ R_{I2} = \frac{A_{I2}}{A_S}$$

It is apparent that the terms involving R_{I1} and R_{I2} can be eliminated when $R_{I1,2} << 1$. Also, the value of $S_K(t)$ may permit the elimination of the product terms involving $S_K(t)$, R_{I1}, and R_{I2}.

Square Law Detector, Application 6, a Desired AM Signal and an Undesired FM Signal. The previous examples have considered various cases of AM interference to a desired AM signal. A second general category exists in which the interference consists of a single or multiple phase frequency modulated signal. A general frequency modulated interfering signal can be expressed as

$$I_{FM}(t) = A_I \cos\left[(\omega_o + \Delta\omega)t + \sum_{K=1}^{\infty} B_K \sin \omega_{IK} t\right] \qquad \text{(I-47a)}$$

$$= A_I \cos \left[\Delta\omega t + \omega_o t + I_{Ko}(t)\right] \qquad \text{(I-47b)}$$

$$= A_I \cos \left[\Delta\omega t + I_{Ko}(t)\right] \cos \omega_o t$$
$$- A_I \sin \left[\Delta\omega t + I_{Ko}(t)\right] \sin \omega_o t \qquad \text{(I-47c)}$$

For a desired AM signal the combined output is found to be

$$v(t) = S_{AM}(t) + I_{FM}(t) \qquad \text{(I-48a)}$$
$$= \left\{A_S\left[1 + S_K(t)\right] + A_I \cos \left[\Delta\omega t + I_{Ko}(t)\right]\right\} \cos \omega_o t$$
$$- \left\{A_I \sin\left[\Delta\omega t + I_{Ko}(t)\right]\right\} \sin \omega_o t \qquad \text{(I-48b)}$$

The square law detected output is, therefore

$$\frac{|V(t)|^2}{A_S^2} = \left[1 + S_K(t)\right]^2 + 2 R_I\left[1 + S_K(t)\right] \cos\left[\Delta\omega t\right.$$ (Continued on next page.)

$$+ I_{KO}(t)\Big] + R_I^2 \qquad \text{(I-49a)}$$

$$= 1 + R_I^2 + 2\, S_K(t) + S_K^2(t) + 2\, R_I \Big[1 + S_K(t)\Big] \cos\Big[I_{Ko}(t)\Big] \cos \Delta\omega t - 2R_I\Big[1 + S_K(t)\Big] \sin\Big[I_{Ko}(t)\Big] \sin \Delta\omega t \qquad \text{(I-49b)}$$

The terms $\cos\left[I_{Ko}(t)\right]$ and $\sin\left[I_{Ko}(t)\right]$ can be evaluated in terms of Bessel functions. The evaluation of these functions will be discussed under phase detectors. However, the evaluation for a single sinusoidal component with a value of $B \gg 1$, will result in a large number of terms (roughly 25 to 50 depending upon the approximation desired). It is, therefore, desired to re-examine the general expressions and attempt to obtain simplified solutions.

Due to the presence of the IF bandpass amplifier, the general expression for an AM desired signal plus FM interference is not actually as general as previously indicated. Equation (I-47) is a realistic representation only when the AM and FM signals are unaffected by the IF amplifier. The system under consideration is therefore, one in which the AM is of a much wider bandwidth than the FM.

The most commonly encountered case is, however, one in which the bandwidth of the FM is much wider than that of the AM. This results in AM modulated pulses, due to the FM sweeping the IF bandpass. The shape of the pulses is determined by the bandpass characteristic of the IF. The output signal can approximately be represented by that shown in Figure I-3. This waveform has been arrived at by approximating a gaussian IF bandpass characteristic by the cos x function.* A pulsed cosine train can in turn be represented by

$$I(t) = \frac{A\tau}{\pi T} + \sum_{K=1}^{\infty} \left[\frac{2A\tau\ T}{\pi} \left(\frac{\cos\ \pi\ K\tau/2T}{T^2 - K^2\tau^2} \right) \right] \cos \frac{2\pi K t}{\tau} \qquad \text{(I-50)}$$

where,

T = pulse period.

$\tau/2$ = pulse width.

A = pulse amplitude.

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*See reference (10) for a discussion of this approximation.

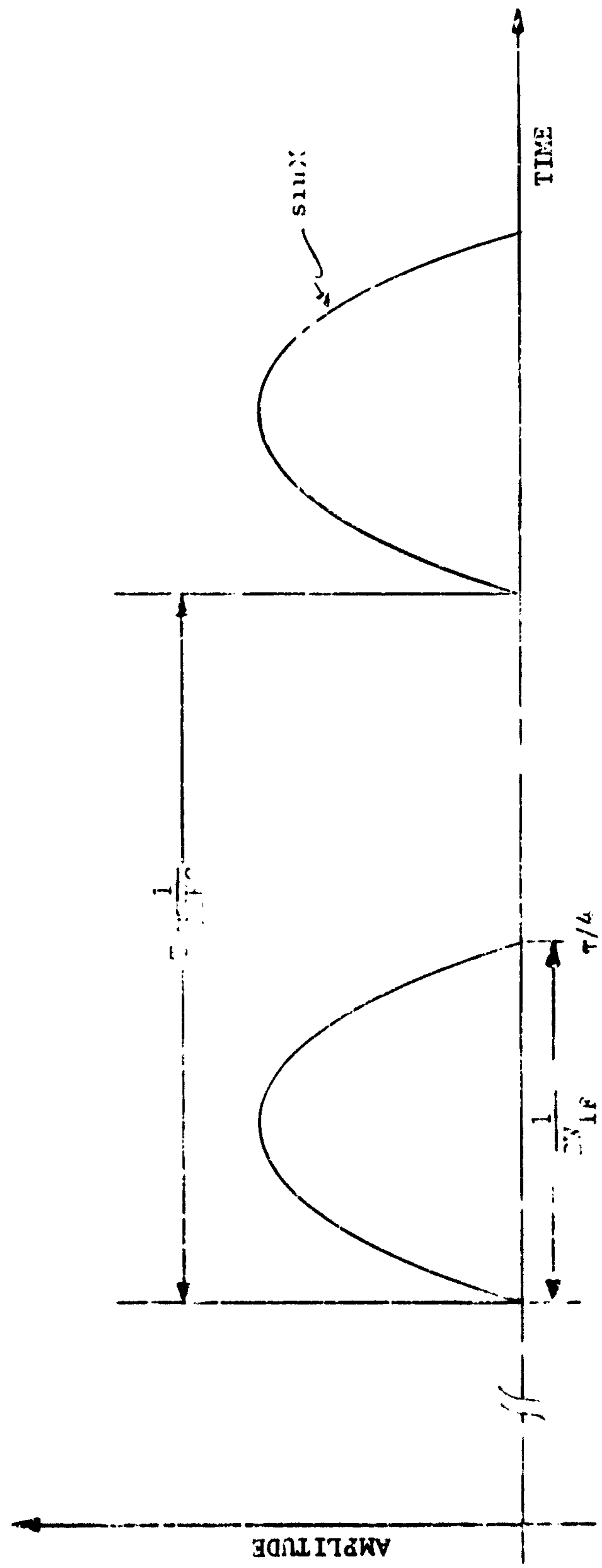

Figure I-3. Cos X Pulse Approximation.

For any particular set of parameters, a finite number of terms can be used to approximate the desired pulse train. For the particular case of commercial FM and a worst frequency type of problem,

where,

$$T = \frac{1}{75\ \text{kc}}$$

and,

$$T >> \tau$$

assume for convenience that

$$T = 10\tau$$

Substituting these values into equation (I-50) the following approximation was obtained (see Figure I-4)

$$I_{\substack{\text{cosine}\\ \text{pulse}}}(t) \approx \sum_{K=1}^{4} \left[.998 \cos \frac{2\pi t}{T} + .996 \cos \frac{\pi t}{T} + .991 \cos \frac{2\pi t}{3T} + .984 \cos \frac{\pi t}{2T}\right] \tag{I-51}$$

It is, therefore, only necessary to substitute the value of the constant coefficient into equation (I-43) to obtain the detected output signal for the isochronous cases ($\Delta\omega = 0$). It is also possible to readily extend these results to the general off-tuned case. It can be readily shown that for the off-tuned case

$$\frac{I_{\substack{\text{off tune}\\ \text{cos pulse}}}}{A_S^{2}} = I_{CP}(t)\left\{R_I\left[1 + B_I \sin \omega_I t\right] \cos\left[B_I \sin \omega_I t\right] \cos n\Delta\omega t - R_I \left[1 + B_I \sin \omega_I t\right] \sin\left[B_I \sin \omega_I t\right] \sin n\Delta\omega t\right\} \tag{I-52}$$

<u>Square Law Detector, Application 7, a Desired AM Signal and an Undesired SSB Signal</u>. Consider the general SSB interference to the general AM square law detected signal. The desired and undesired signal are given by

$$s(t) = A_S \left[1 + S_K(t)\right] \cos \omega_o t \tag{I-53}$$

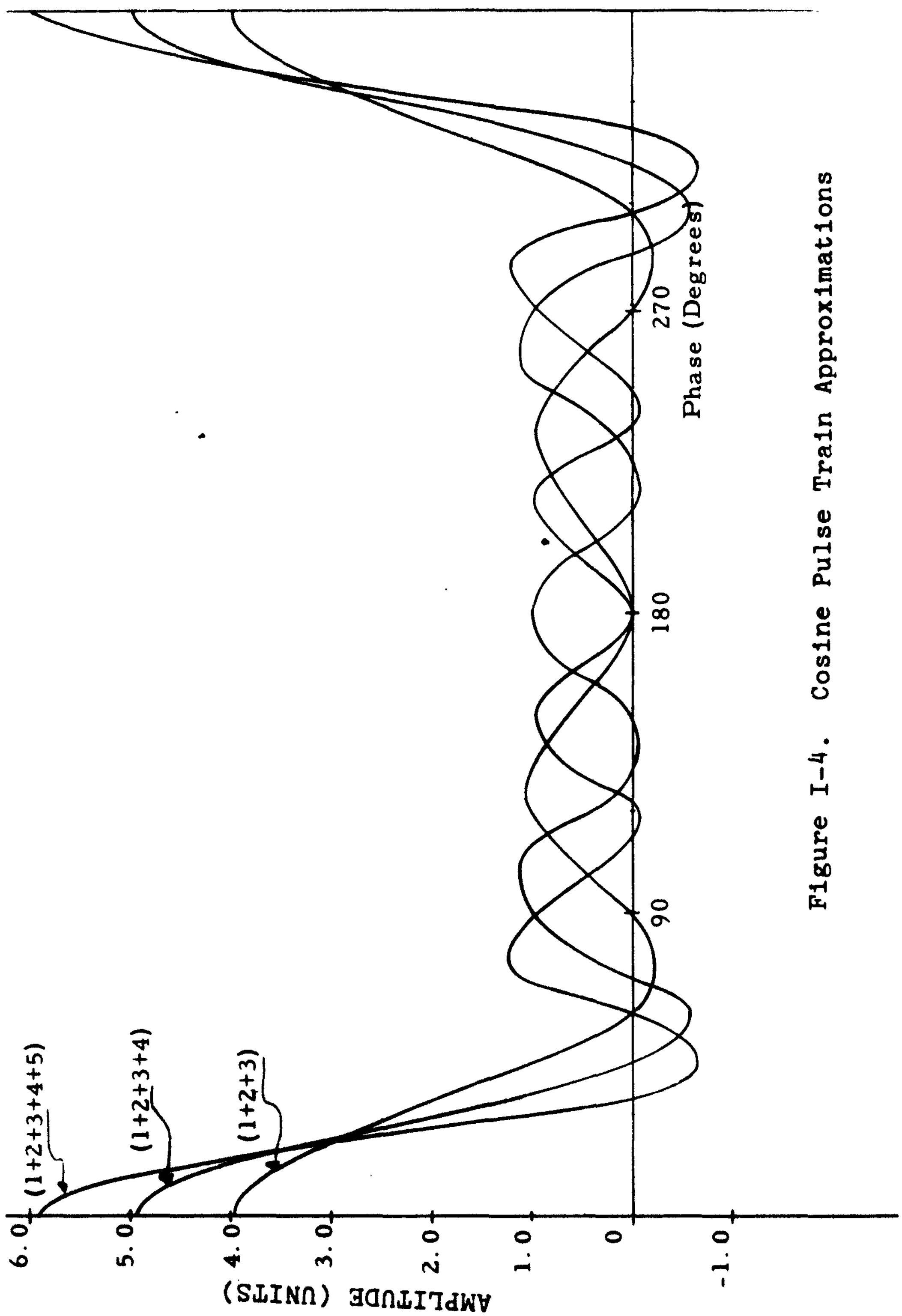

Figure I-4. Cosine Pulse Train Approximations

$$I(t) = A_I \sum m_{IK} \cos\left[(\omega_o + \Delta\omega + \omega_{IK})\, t + \theta_I\right] \tag{I-54a}$$

$$\begin{aligned} &= A_I\Big[I_{KE}(t) \cos\,(\Delta\omega t + \theta_I) \\ &- I_{Ko}(t) \sin\,(\Delta\omega t + \theta_I)\Big] \cos\,\omega_o t \\ &- A_I\Big[I_{Ko}(t) \cos\,(\Delta\omega t + \theta_I) \\ &+ I_{KE}(t) \sin\,(\Delta\omega t + \theta_I)\Big] \sin\,\omega_o t \end{aligned} \tag{I-54b}$$

The square law detector output becomes

$$\begin{aligned} \frac{|V(t)|^2}{A_S{}^2} = &\left[1 + S_K(t)\right]^2 + R_I^2 \Big[S_K(t) \cos\,(\Delta\omega t + \theta_I) \\ &+ S_{Ko}(t) \sin\,(\Delta\omega t + \theta_I)\Big]^2 \\ &+ R_I\left[1 + S_K(t)\right] \Big[S_K(t) \cos\,(\Delta\omega t + \theta_I) \\ &+ S_{Ko}(t) \sin\,(\Delta\omega t + \theta_I)\Big] \\ &+ R_I^2\Big[S_K(t) \sin\,(\Delta\omega t + \theta_I) \\ &+ S_{Ko}(t) \cos\,(\Delta\omega t + \theta_I)\Big]^2 \end{aligned} \tag{I-55a}$$

$$\begin{aligned} &= R_I{}^2 + \left[1 + S_K(t)\right]^2 + R_I \Big[1 \\ &+ S_K(t)\Big] \Big\{I_{KE}(t) \cos\,(\Delta\omega t + \theta_I) \\ &+ I_{Ko}(t) \sin\,(\Delta\omega t + \theta_I)\Big\} \end{aligned} \tag{I-55b}$$

For the general case of an unsuppressed carrier SSB interference to an AM square law case we only need to add the terms

$$I_{carrier}(t) = A_I \cos\,\left[(\omega_o + \Delta\omega)t + \theta_I\right] \tag{I-56a}$$

$$\begin{aligned} &= \left[A_I \cos\,(\Delta\omega t + \theta_I)\right] \cos\,\omega_o t \\ &- \left[A_I \sin\,(\Delta\omega t + \theta_I)\right] \sin\,\omega_o t \end{aligned} \tag{I-56b}$$

to equation (I-55b).

Therefore the answer can be obtained from the previous results and the unnormalized answer shown to be

$$|V(t)|^2 = \Big\{A_I\left[1 + I_{KE}(t)\right] \cos\,(\Delta\omega t + \theta_I)$$ (Continued on next page)

$$- A_I I_{Ko}(t) \sin (\Delta\omega t + \theta_I)$$
$$+ A_S\left[1 + S_K(t)\right]\Big\}^2 + A_I^2\Big\{\Big[1$$
$$+ I_{Ko}(t)\Big] \cos (\Delta\omega t + \theta_I)$$
$$+ I_{KE}(t) \sin (\Delta\omega t + \theta_I)\Big]\Big\}^2 \qquad \text{(I-57)}$$

This form can be considerably simplified depending upon whether $I_K(t)$ is an even or odd function.

LINEAR DETECTOR

The linear detector output can be obtained from the previously derived unfiltered square law outputs by computing the absolute value of the square root of those results. That is, we should obtain

$$|V(t)| = \left|\begin{bmatrix}\text{square law}\\ \text{output}\end{bmatrix}^{1/2}\right| \qquad \text{(I-58)}$$

The square law output has been previously shown to be, in general, quite complicated and consists of a large number of non-multiple cosine terms. The problem of obtaining the square root of a number of cosine terms appears difficult to solve. This problem is discussed in reference (11). The transform method of analysis is used in this discussion and it is shown that, for this method, solutions can be obtained only for the cases of two sinusoidal terms. For the cases of three or more sinusoidal terms, and no constants, it is possible to approximate the answer with increasing difficulty. Due to this increasing difficulty and the fact that the answers are approximations, it may be more reasonable to try other methods of solution. In particular, if the time-amplitude output is desired, a numerical approximation method seems more reasonable, see reference (12). If, on the other hand, only the output power spectrum is desired, the correlation function approach may be easier to handle. This particular approach is also discussed in reference (11).

Within the bounds of the present discussion, only the more finite closed form solutions to the linear detector will be discussed. Perhaps the easiest approach to the general problem is to show the solutions to problems of gradually increasing complexity.

Linear Detector, Application 1, a Desired and Undesired CW Signal. The simplest interference problem is that of an off-tuned interfering carrier with a desired carrier. The desired solution is found from equation (I-20b). After setting $m_s = 0$ and normal-

izing, the solution is

$$\frac{|V(t)|}{A_S} = \left[1 + R_I^2 + 2R_I \cos(\Delta\omega t + \theta_I)\right]^{\frac{1}{2}} \tag{I-59a}$$

$$= \left[1 + R_I^2 + 2R_I \cos\theta\right]^{\frac{1}{2}} \tag{I-59b}$$

where

$$R_I = A_I/A_S, \quad \Delta\omega t + \theta_I = \theta$$

This can be rewritten as

$$\frac{|V(t)|}{A_S} = \frac{1}{A_S} \quad \frac{|V(t)|^2}{|V(t)|} \tag{I-60}$$

where,

the quantity $|V(t)|^{-1}$ can be expanded using a Legendre series

$$|V(t)|^{-1} = \left[1 + 2 R_I \cos\theta + R_I^2\right]^{-\frac{1}{2}} \tag{I-61a}$$

$$|V(t)|^{-1} = P_o(\cos\theta) + R_I P_I(\cos\theta) + \ldots\ldots R_I^q P_q(\cos\theta) \tag{I-61b}$$

when

$$R_I < 1$$

The above expression can be further simplified and has been shown by Aiken (13) to be of the form

$$\frac{|V(t)|}{A_S} = 1 + R_I \cos\theta + R_I \sin\theta \left[\frac{R_I P_1'}{1.2} + \frac{R_I^2}{2.3} P_2' + \ldots\ldots \frac{R_I^q P_q'}{q(q+1)}\right] \tag{I-62}$$

where P_1', P_2' ...P_q' are given in Jahnke and Emde, reference (14) page 124.

When $R_I \ll 1$, a simpler approach to the solution of equation (I-59b) is to use the binomial theorem. Rewriting this equation as

$$\frac{|V(t)|}{A_S} = \left[(1 + R_I)^2 + 2\, R_I^2(\cos \Delta\omega t - 1)\right]^{\frac{1}{2}} \tag{I-63}$$

This can be expanded using the binomial theorem (see equation I-88). The first three terms are obtained as

$$\frac{|V(t)|}{A_S} \approx (1 + R_I) + \frac{R_I^2(\cos \Delta\omega t - 1)}{(1 + R_I)} - \frac{R_I^4 (\cos \Delta\omega t - 1)^2}{8\,(1 + R_I)^3} + \ldots, \tag{I-64}$$

which is again approximately

$$\frac{|V(t)|}{A_S} \approx 1 + R_I \cos (\Delta\omega t + \theta_I) \tag{I-65}$$

when $R_I \ll 1$

<u>Linear Detector, Application 2, A Desired and Undesired Tone Modulated AM Signal</u>. A more complicated interference problem is one with modulated sidebands where the signal is expressed by

$$I_{AM}(t) = A_I\,(1 + m_I \cos \omega_I t) \cos (\omega_o + \Delta\omega)t \tag{I-66}$$

$$S_{AM}(t) = A_S\,(1 + m_S \cos \omega_S t) \cos \omega_o t \tag{I-67}$$

The constant R_I in equation (I-61) becomes

$$R_I' = \frac{A_I}{A_S}\,\frac{(1 + m_I \cos \omega_I t)}{(1 + m_S \cos \omega_S t)} = R_I\,\frac{(1 + m_I \cos \omega_I t)}{(1 + m_S \cos \omega_S t)} \tag{I-68}$$

and it is necessary to again restrict R_I' to a value less than 1.

The worst case that can be allowed is

$$\frac{A_I\,(1 + m_I)}{A_S\,(1 - m_S)} < 1 \tag{I-69}$$

If the above restriction is met, the output can be obtained by expanding R_I. R_I' contains component frequencies in the deno-

minator and it is necessary to expand this in the Fourier series

$$\frac{1}{(1 + m \cos \omega_I t)^q} = \frac{a_{qo}}{2} + a_{q1} \cos \omega_I t + a_{q2} \cos 2 \omega_I t \qquad \text{(I-70)}$$

$$+ \ldots\ldots a_{qn} \cos n \omega_I t$$

which can be evaluated with the aid of reference (15), table 66, No 5. After rearranging coefficients and changing to the present nomenclature the answer was shown by Aiken (see equation I-62 above) to be

$$\frac{|V(t)|}{A_S} = 1 + \left[\frac{R_I^2 (2 + m_I^2)}{8 (1 - m_S^2)^{1/2}} + \frac{R_I^4(1 + 3m_I^2 + 3m_I^4/8)(2 + m_S^2)}{128 (1 - m_S^4)^{5/2}} + \ldots\right]$$

$$\pm \left[m_S - \frac{m_S R_I^2 (2 + m_I^2)}{4\left[(1-m_S^2)+(1-m_S^2)^{1/2}\right]} - \frac{3m_S R_I^4(1 + 3m_I^2 + 3m_I^4/8)}{64 (1-m_S^2)^{5/2}} + \ldots\right] \cos \omega_S t$$

$$+\left[\frac{m_I R_I^2}{2(1-m_S^2)^{1/2}} + \frac{m_I R_I^4(1 + 3m_I^2/4)(2 + m_S^2)}{32 (1 - m_S)^{5/2}} + \ldots\right] \cos \omega_I t$$

$$+\left[R_I - \frac{R_I^3 (2 + 3m_I^2)}{16 (1-m_S^2)^{3/2}} + \ldots\right] \cos \Delta\omega t$$

$$+\left[\frac{m_I^2 R_I^2}{8 (1-m_S^2)^{1/2}} + \frac{m_I^2 R_I^4(3 + m_I^2/2)(2 + m_S^2)}{128 (1 - m_S^2)^{5/2}} + \ldots\right] \cos 2\omega_I t$$

$$+\left[\frac{R_I^2(2 + m_I^2)}{4 (1-m_S^2)^{1/2}} - \frac{R_I^4 (1 + 3m_I + 3m_I^4/8)(2 + m_S^2)}{32 (1 - m_S)^{5/2}} + \ldots\right] \cos 2\Delta\omega t$$

(Continued on next page)

$$+\left[\frac{m_I R_I}{2} - \frac{R_I^{\,3}\ 3\ (4 + m_I^{\,2})}{64\ (1 - m_S^{\,2})^{3/2}} + \dots\right] \cos\ (\omega_I \pm \Delta\omega)t$$

$$+\left[\frac{m_S R_I^{\,3}\ (2 + 3\ m_I^{\,2})}{16\ (1 - m_S^{\,2})^{3/2}} + \dots\right] \cos\ (\omega_S \pm \Delta\omega)t + \dots \qquad \text{(I-71a)}$$

For small values of R_I this same answer can also be obtained from the binominal theorem and is also given by Aiken. It is convenient to reduce the large expression given by Aiken for either deviation by dropping carrier ratios higher than the second order. The resultant normalized answer is

$$\frac{|V(t)|}{A_S} \approx 1 + m_S \cos \omega_S t + \frac{m_I R_I^{\,2}}{2} \cos \omega_I t + R_I \cos \Delta\omega t$$

$$+ \frac{R_I^{\,2}}{4} \cos 2\ \Delta\omega t + \frac{m_S R_I^{\,3}}{8} \cos\ (\omega_S \pm \Delta\omega)t$$

$$+ \frac{m_I R_I}{2} \cos\ (\omega_I \pm \Delta\omega)t + \frac{m_I^{\,2} R_I^{\,2}}{8} \cos 2\omega_I t \qquad \text{(I-71b)}$$

Using either equation (I-71a) or (I-71b) the amplitude of a desired harmonic can be obtained. In reference(16) the total interference power and the carrier beat power are plotted as a function of the desired signal power and the carrier amplitude ratio, R_I. These results also indicate that for a wide range of parameters the beat tone contributes the majority of the interference power.*

The previous problem could be generalized still further, since we could have assumed a complex modulated signal, expressed as

$$I_{AM}(t) = A_I\left[1 + I_K(t)\right] \cos\ (\omega_o + \Delta\omega)t \qquad \text{(I-72)}$$

and,

$$S_{AM}(t) = A_S\left[1 + S_K(t)\right] \cos\ \omega_o t \qquad \text{(I-73)}$$

The expressions previously shown are still valid with the restriction that

*but not necessarily the majority of the degradation!

$$R_I \frac{\left[1 + I_K(t)\right]}{\left[1 - S_K(t)\right]} = R_I'' < 1 \tag{I-74}$$

It is apparent that the average modulation indexes must be extremely small for this to be valid since, again, in the worst case

$$R_I \frac{\left(1 + \sum_{K=1}^{\infty} m_{IK}\right)}{\left(1 - \sum_{K=1}^{\infty} m_{SK}\right)} < 1 \tag{I-75}$$

This method is therefore, in general impractical for complex modulated signals due to the unrealistically small modulation coefficients required. Any particular combination of signals that is chosen that meets or approximately meets this restriction can be handled by this method. If the more general interfering signal is assumed

$$I(t) = A_{I1} \cos (\omega_o + \Delta\omega)t + A_{I2} \cos (\omega_o + \Delta\omega_2)t$$

$$+ .. A_{IN} \cos (\omega_o + \Delta\omega_N)t \tag{I-76}$$

where the A_{IN} include or do not include modulation. The linear detector output then becomes one of obtaining solutions to an equation of the form

$$\frac{|V(t)|}{A_o} = \left[A_o + \frac{A_1}{A_o} \cos \omega_1 t + \frac{A_2}{A_o} \cos \omega_2 t + \ldots \frac{A_N}{A_o} \cos \omega_N t\right]^{\frac{1}{2}} \qquad \text{(I-77}$$

A_o = normalizing amplitude

$\omega_1, \omega_2, \ldots \omega_N$ are, in general, incommensurable.

This problem is discussed further in reference(11), Appendix B, where an attempt to use the transform method (in many cases a more powerful method than the straightforward approach) is also limited to the case of two input carriers of different frequencies.

It, therefore, appears from this study that complete sol-

utions beyond this point become one of making appropriate numerical approximations.

It is still possible, however, to make some reasonable approximation for the case in which $A_S >> A_I$ or $A_I >> A_S$.

Linear Detector, Application 3, Approximations for a Desired and Undesired AM Signal. It is now desired to consider small interfering signal approximations. From the discussion of Application 2 of the square law detector, it would be desirous to obtain the linear detector output of the general AM interference case. From equation (I-35b) we obtain the linear detector output

$$|V(t)| = \Big\{A_S^2 + A_I^2 + 2\left[A_S^2 S_K(t) + A_I^2 I_K(t)\right] + A_S^2 S_K^2(t) + A_I^2 I_K^2(t) + 2 A_S A_I \cos(\Delta\omega t + \theta_I)\left[1 + S_K(t) + I_K(t) + S_K(t) I_K(t)\right]\Big\}^{\frac{1}{2}} \quad \text{(I-78)}$$

For the important case in which $A_S > A$,

$$\frac{|V(t)|}{A_S} = \Big\{1 + R_I^2 + 2\left[S_K(t) + R_I^2 I_K(t)\right] + S_K^2(t) + R_I^2 I_K^2(t) + 2 R_I \cos(\Delta\omega t + \theta_I)\left[1 + S_K(t) + I_K(t) + S_K(t) I_K(t)\right]\Big\}^{1/2} \quad \text{(I-79)}$$

When we also have that $R_I << 1$,

$$\frac{|V(t)|}{A_S} \approx \Big\{1 + 2S_K(t) + R_I^2 I_K(t) + S_K^2(t) + 2R_I \cos(\Delta\omega t + \theta_I)\left[1 + S_K(t) + I_K(t) + S_K(t) I_K(t)\right]\Big\}^{\frac{1}{2}} \quad \text{(I-80)}$$

Consider the special case of equation (I-64) in which $I(t) = 0$. The output is therefore given by

$$\frac{|V(t)|}{A_S} = \Big\{1 + 2 S_K(t) + S_K^2(t)\Big\}^{\frac{1}{2}} = 1 + S_K(t) \quad \text{(I-81)}$$

It is apparent that the interfering terms introduce error terms in the completion of the square and it is convenient to rewrite equation (I-64) in the form

$$\frac{|V(t)|}{A_S} = \left\{\left[1 + S_K(t)\right]^2 + E_I(t)\right\}^{\frac{1}{2}} = \left\{1 + f(t)\right\}^{\frac{1}{2}} \tag{I-82}$$

This can be expanded in a binomial series (see equation I-88) as long as

$$f^2(t) < 1$$

When

$$2\, S_K(t) + S_K{}^2(t) + E_I(t) << 1$$

the first three terms of the series can be used and we obtain

$$\frac{|V(t)|}{A_S} \approx 1 + \left[S_K(t) + \frac{S_K(t)}{2} + \frac{E_I(t)}{2}\right] - \left[\frac{S_K{}^2(t)}{2} + \frac{S_K{}^4(t)}{8} + \frac{E_I{}^2(t)}{8} + \frac{S_K{}^3(t)}{2} + \frac{S_K(t)\, E_I(t)}{2} + \frac{S_K{}^2(t)\, E_I(t)}{4}\right] \tag{I-83a}$$

$$\approx 1 + S_K(t) + E_R(t) \tag{I-83b}$$

where

$$E_R(t) = 3/8\; E_I(t) - \frac{E_I(t)\, S_K(t)}{2} - \frac{E_I(t)\, S_K{}^2(t)}{4} - \frac{S_K{}^3(t)}{2} - \frac{S_K{}^4(t)}{8}$$

The first order interference can therefore be represented as

$$E_R(t) \approx 3/8\; E_I(t) \tag{I-84}$$

which results in the desired output as

$$\frac{|V(t)|}{A_S} \approx 1 + S_K(t) + .75\, R_I{}^2 I_K(t) + .75 R_I \cos\,(\Delta\omega t + \theta_I) \left[1 + S_K(t) + I_K(t) + S_K(t)\, I_K(t)\right] \tag{I-85}$$

This is restricted to the case in which

$$R_I << 1 \text{ and } f^2(t) < 1$$

The constant in front of R cos $\Delta\omega t$ has, because of this approximation, changed from 1 in equations (I-65) and (I-71b) to .75. This shows that, to be strictly correct, more terms in the expansion should be used. This would in turn complicate the answer so that the previous equation can be used realizing that it is an approximation.

A more detailed derivation of the series method can be accomplished by considering the case in which there is no interference modulation and only a general AM desired signal and an undesired off-tuned carrier. From (I-59a) we have

$$|V(t)| = \left[A_S^2(t) + A_I^2 + 2A_IA_S(t) \cos \phi\right]^{1/2} \tag{I-86}$$

where

$$\phi = \Delta\omega t + \theta_I$$

Completing the square we obtain

$$|V(t)| = \left[A_S^2(t) + 2A_S(t)A_I + A_I^2 - 2A_S(t)A_I \ (1-\cos\phi)\right]^{1/2} \tag{I-87a}$$

$$= A_S(t) + A_I \left\{1 - \frac{4R_S(t)}{\left[1 + R_S(t)\right]^2} \sin^2 \phi/2\right\}^{1/2} \tag{I-87b}$$

The square root operation can be evaluated by use of the complete series.

$$\sqrt{1 - X} = \left\{1 - \frac{X}{2} - \sum_{N=2}^{\infty} \frac{4\ (2N-3)!}{n!(n-2)!} \left(\frac{X}{4}\right)^n\right\}; \ x < 1 \tag{I-88}$$

Performing the indicated operation we obtain

$$|V(t)| = A_S(t) + A_I \left\{\frac{1}{1 + R_S(t)} + \frac{R_S(t)}{1 + R_S(t)} \cos \phi - \right.$$

$$\left. \sum_{n=2}^{\infty} \frac{4(2n - 3)!}{n!(n - 2)!} \left[\frac{R_S{}^n(t)}{[1 + R_S(t)]^{2n-1}}\right] \sin^{2n} \frac{\phi}{2}\right\} \tag{I-89}$$

This representation is completely applicable except for the singular

condition $\{R_S(t) = 1, \phi = k\pi\}$ where k is any odd integer. The detector output signal can now be expressed in the form

$$|V(t)| = S_o(t) + I_o(t) \tag{I-90}$$

where

$$I_o(t) \; A_I \left\{ \frac{1}{1 + R_t} + \frac{R_t}{1 + R_t} \cos \phi - \sum_{n=2}^{\infty} \frac{4(2n - 3)!}{n!(n - 2)!} \left[\frac{R_t^{\,n}}{(1+R_t)^{2n-1}} \right] \sin^{2n} \frac{1}{2} \phi \right\} m_I(t) \tag{I-91}$$

where

$R_S(t)$ has been changed tc R_t for symbolic convienence.

Thus, we have expressed $|V(t)|$ as the sum of the desired demodulated signal $S_o(t) \equiv \left[A_S \; (1 + m_S(t)\;)\right]$ and an interference signal $I_o(t)$.

We now wish to develop a more reasonable form for I_t. By inspection of (I-91) it can be seen that I_t is a Fourier series in ϕ, i.e.,

$$I_t = I \sum_{p=o}^{\infty} A_p \; (R_t) \cos p\phi \qquad (m_I = 1) \tag{I-92}$$

where

$$A_p(R_t) \equiv \begin{cases} \dfrac{2}{\pi} \displaystyle\int_o^{\pi} \frac{I_t}{A_I} \cos p\phi \; d\phi & (P \neq 0) \\[2ex] \dfrac{1}{\pi} \displaystyle\int_o^{\pi} \frac{I_t}{A_I} \; d\phi & (p = 0) \end{cases} \tag{I-93}$$

It is thus possible to find $A_p(R_t)$ for all p by combining (I-93) with (I-91). We shall confine ourselves here to obtaining the three most important coefficients of the series, A_0, A_1, and A_2. To do this, we make the substitution

$$\frac{1}{2} \phi \equiv \psi \tag{I-94}$$

and use the trigonometric identities

$$\cos\phi \cos 2\psi = 1 - 2\sin^2\psi \tag{I-95}$$

From equation (I-93), (I-95), and (I-96), we can derive A_0, A_1, A_2 from the following expressions:

$$\cos 2\phi = \cos 4\psi = 1 - 8\sin^2\psi + 8\sin^4\psi \tag{I-96}$$

$$A_o = \frac{1}{1+R_t} - \sum_{n=2}^{\infty} \frac{4(2n-3)!}{n!(n-2)!}\left[\frac{R_t^{\,n}}{(1+R_t)^{2n-1}}\right]\cdot \frac{1}{\pi}\int_0^{\pi} \sin^{2n}\psi\, d\psi \tag{I-97}$$

$$A_1 = \frac{R_t}{1+R_t} - \sum_{n=2}^{\infty} \frac{4(2n-3)!}{n!(n-2)!}\left[\frac{R_t^{\,n}}{(1+R_t)^{2n-1}}\right]\cdot \frac{2}{\pi}\int_0^{\pi} \sin^{2n}\psi\,(1 - 2\sin^2\psi)\,d\psi \tag{I-98}$$

and

$$A_2 = -\sum_{n=2}^{\infty} \frac{4(2n-3)!}{n!(n-2)!}\left[\frac{R_t}{(1+R_t)^{2n-1}}\right]\frac{2}{\pi}\int_0^{\pi} \sin^{2n}\psi(1-8\sin^2\psi+8\sin^4\psi)\,d\psi \tag{I-99}$$

These expressions can all be reduced by use of the relationship

$$\frac{1}{\pi}\int_0^{\pi} \sin^{2k}\psi\, d\psi = \frac{(2k)!}{\{2^k k!\}^2}, \quad k = 1, 2, 3, \text{etc.} \tag{I-100}$$

By straightforward manipulation we thus obtain A_0, A_1, and A_2 in the forms

$$Ao = \left\{\frac{1}{1+R_t} - \sum_{n=2}^{\infty} \frac{(2n-3)!(2n)!}{4^{n-1}(n!)^3(n-2)!}\left[\frac{R_t^{\,n}}{(1+R_t)^{2n-1}}\right]\right\} \tag{I-101}$$

$$A_1 = \left\{\frac{R_t}{1+R_t} + \sum_{n=2}^{\infty} \frac{(2n-3)!(2n)!}{4^{2n-1}(n!)^3(n-2)!}\,\frac{2n}{n+1}\left[\frac{R_t^{\,n}}{(1+R_t)^{2n-1}}\right]\right\} \tag{I-102}$$

$$A_2 = -\sum \frac{(2n-3)!\,(2n)!}{\mu^{2n-1}(n!)^3(n-2)!}\;\frac{2n}{n+1}\;\frac{n-1}{n+2}\left[\frac{R_t^n}{(1+R_t)^{2n-1}}\right] \qquad \text{(I-103)}$$

The evaluation is still not complete since these terms are still a function of $R_t = R_s(t)$. For small values of $m_s(t)$, and perhaps in general, it is most convenient to use the approximation

$$R_S(t) = R_S\left[1 + \overline{m_S^2(t)}\right]^{\frac{1}{2}} \qquad \text{(I-104)}$$

The series terms can then be evaluated with the aid of equations (I-101) through (I-104). This method (in contrast to the previous series method) obtains a more accurate constant at the loss of generality in the interfering term.

The first series method can also be evaluated for the case when the interference is the strongest signal.

For the case when $A_I >> A_S$, it is convenient to write equation (I-78) in the form

$$\frac{|V(t)|}{A_I} = \{R_S^2 + 1 + 2\,R_S^2\,S_K(t) + 2\,I_K(t) + R_I^2 S_K^2(t) + I_K^2(t) + 2\,R_S \cos\Delta\omega t\left[1 + S_K(t) + I_K(t) + S_K(t)\,I_K(t)\right]\}^{\frac{1}{2}} \qquad \text{(I-105)}$$

where,

$$R_S = A_S/A_I.$$

When

$$R_S << 1,$$

$$\frac{|V(t)|}{A_I} \simeq \{1 + 2\,R_S^2 S_K(t) + 2\,I_K(t) + I_K^2(t)\}^{\frac{1}{2}} \qquad \text{(I-106)}$$

which is from the previous arguments approximately equal to (taking the first two terms):

$$\frac{|V(t)|}{A_I} \simeq 1 + \left\{R_S^2\,S_K(t) + I_K(t) + \frac{I_K^2(t)}{2}\right\} - \left\{\frac{R_S^4\,S_K^2(t)}{2} + \frac{I_K^2(t)}{2} + \frac{I_K^4(t)}{8} + R_S^2\,S_K(t) + I_K(t)\right.$$

(Continued on next page)

$$+ \frac{I_K^3(t)}{2} + \frac{R_I^2\ S_K(t)\ I_K^2(t)}{2} \Bigg\} \qquad \text{(I-107)}$$

which can further be simplified to

$$\frac{|V(t)|}{A_I} \approx 1 + R_S^2\ S_K(t) + I_K(t) \qquad \text{(I-108)}$$

where only the gross first order interference effects are considered.

<u>Linear Detector, Application 4, Approximations for a Desired CW Signal and an Undesired Pulsed Signal</u>. Consider the problem of a pulsed interfering signal interfering with a desired unmodulated carrier. The approach adopted will, again, be to consider approximations for large and small carrier ratios.

For this case the signals can be written as

$$S(t) = A_S \cos \omega_o t \qquad \text{(I-109)}$$

$$I(t) = A_I A_{IN}(t) \cos \left[(\omega_o + \Delta\omega)t + \theta_I\right] \qquad \text{(I-110)}$$

where $A_{IN}(t)$ = the normalized pulsed description. The combined signal is given by

$$V(t) = S(t) + I(t) \qquad \text{(I-111a)}$$

$$= \left[A_S^2 + A_I^2\ A_{IN}^2(t) + 2\ A_S A_I A_{IN}(t) \cos (\Delta\omega t + \theta_I)\right]^{\frac{1}{2}} \times \cos (\omega_o t + \phi) \qquad \text{(I-111b)}$$

For the case in which $A_S^2 >> A_I^2$

$$V(t) \approx \left[A_S^2 + 2 A_S A_I A_{IN}(t) \cos (\Delta\omega t + \theta_I)\right]^{\frac{1}{2}} \cos (\omega_o t + \phi) \qquad \text{(I-112)}$$

where the second order term has been dropped.

The linear detector output can again approximately be found from the binomial series as:

$$\frac{|V(t)|}{A_S} = 1 + R_I A_{IN}(t) \cos (\Delta\omega t + \theta_I)$$ (Continued on next page.)

$$- \frac{1}{2} R_I{}^2 A_{IN}{}^2(t) \cos{}^2(\Delta\omega t + \theta_I)$$

$$+ \frac{1}{2} R_I{}^3 A_{IN}{}^3(t) \cos{}^3(\Delta\omega t + \theta_I) \qquad \text{(I-113a)}$$

$$= 1 + \frac{R_I{}^2 A_{IN}{}^2(t)}{4}$$

$$+ \left[R_I\, A_{IN}(t) + \frac{R_I{}^3 A_{IN}(t)}{4} + \ldots \right] \cos (\Delta\omega t + \theta_I)$$

$$- \left[\frac{R_I{}^2 A_{IN}{}^2(t)}{4} + \ldots \right] \cos (2\Delta\omega t + 2\theta_I)$$

$$+ \left[\frac{R_I{}^3 A_{IN}^3(t)}{8} + \ldots \right] \cos (3\Delta\omega t + 3\theta_I)$$

$$- \ldots \qquad \text{(I-113b)}$$

As a second part of this problem consider the case when $A_I{}^2 >> A_S{}^2$. Then, from the previous,

$$|V(t)| = \left[A_S{}^2 + A_I{}^2 A_{IN}^2(t) \right]^{\frac{1}{2}} \qquad \text{(I-114)}$$

Using the same series approach:

$$|V(t)| = A_S + \frac{A_I{}^2 A_{IN}{}^2(t)}{2\, A_S} - \frac{A_I{}^6 A_{IN}{}^6(t)}{8\, A_S{}^3} \qquad \text{(I-115a)}$$

and normalizing,

$$\frac{|V(t)|}{A_S} = 1 + \frac{R_I{}^2 A_{IN}{}^2(t)}{2} - \frac{A_I{}^2 R_I{}^4 A_{IN}{}^6}{8} \qquad \text{(I-115b)}$$

when $A_I{}^2 >> A_S{}^2$.

<u>Linear Detector, Application 5, Approximations for a Desired AM Signal and an Undesired FM Signal</u>. As the next example, consider the case of wide band AM interfered with by FM for the special case when $A_S >> A_I$.

The desired and undesired signals are found to be

$$S(t) = A_S\,(1 + m_S \cos \omega_S t) \cos \omega_o t \tag{I-116}$$

$$I(t) = A_I \cos\left[(\omega_o + \Delta\omega)t + B_I \sin \omega_I t\right] \tag{I-117}$$

From the combined signal the linear detected output is consequently found to be

$$\frac{|V(t)|}{A_S} = \Big[1 + R_I^2 + 2S_K(t) + S_K^2(t) + 2R_I\left[1 + S_K(t)\right]\cos\left[I_{Ko}(t)\right]\cos\Delta\omega t$$
$$- 2R_I\left[1 + S_K(t)\right]\sin\left[I_{Ko}(t)\right]\sin\Delta\omega t\Big]^{1/2} \tag{I-118}$$

If $R_I << 1$, this can approximately be reduced to

$$\frac{|V(t)|}{A_S} = \left\{\left[1 + S_K(t)\right]^2 + E_I(t)\right\}^{1/2} \tag{I-119}$$

where

$$E_I(t) = 2\,R_I\left[1 + S_K(t)\right]\left[\cos A \cos B - \sin A \sin B\right]$$
$$+ 2\,R_I\left[1 + S_K(t)\right]\cos\left[\Delta\omega t + I_{Ko}(t)\right] \tag{I-120}$$

In a similar manner to equation (I-85) we therefore obtain

$$\frac{|V(t)|}{A_S} \approx 1 + S_K(t) + .75\,R_I\left[1 + S_K(t)\right]\cos\left[\Delta\omega t + I_{Ko}(t)\right] \tag{I-121}$$

When $A_I >> A_S$, the same procedure can be used. From equation (I-49a) we obtain

$$|V(t)|^2 = A_S^2\left[1 + S_K(t)\right]^2 + 2A_SA_I\left[1 + S_K(t)\right]\cos\Big[\Delta\omega t$$
$$+ I_{Ko}(t)\Big] + A_I^2 \tag{I-122}$$

For the case in which $A_I >> A_S$ and a linear detector is used:

$$\frac{|V(t)|}{A_I} = \Big[1 + R_S^2\left[1 + S_K(t)\right]^2 + 2R_S\left[1 + S_K(t)\right]\cos\Big[\Delta\omega t$$
$$+ I_{Ko}(t)\Big]\Big]^{1/2} \tag{I-123}$$

When $R_S << 1$

$$\frac{|V(t)|}{A_I} \approx \left[1 + 2\,R_S\left[1 + S_K(t)\right]\cos\left[\Delta\omega t + I_{Ko}(t)\right]\right]^{\frac{1}{2}} \qquad \text{(I-124)}$$

which is approximately equal to using the binomial expansion

$$\frac{|V(t)|}{A_I} \approx 1 - \frac{R_S^{\,2}}{4}\; S_K(t) + R_S\left[1 + S_K(t)\right]\cos\left[\Delta\omega t + I_{Ko}(t)\right] \qquad \text{(I-125)}$$

Linear Detector, Application 6, Approximations for a Desired AM Signal and an Undesired SSB Signal. Consider the case of an AM desired signal interfered with by an undesired SSB signal. From equation (I-55b) the square law output is obtained as

$$|V(t)|^2 = A_I^{\,2} + A_S^{\,2}\left[1 + S_K(t)\right]^2 + 2\,A_S A_I\left[1 + S_K(t)\right]\left[I_K(t)\cos(\Delta\omega t + \theta_I) + I_{Ko}(t)\sin(\Delta\omega t + \theta_I)\right] \qquad \text{(I-126)}$$

For the case when $A_S >> A_I$

$$\frac{|V(t)|}{A_S} \approx 1 + S_K(t) + .75\,R_I\left[1 + S_K(t)\right]\left[I_K(t)\cos\Delta' + I_{Ko}(t)\sin\Delta'\right] \qquad \text{(I-127)}$$

From equation (I-84),

When $A_I >> A_S$

$$\frac{|V(t)|}{A_I} \approx 1 - \frac{R_S^{\,2}}{4}\left[1 + S_K(t)\right]^2 \sum_{K=1}^{N} m_{IK}^{\,2} + R_S\left[1 + S_K(t)\right] \times \sum m_{IK}\cos\left[(\omega_K + \Delta\omega)t + \theta_I\right] + \frac{R_S^{\,2}}{4}\left[1 + S_K(t)\right]^2 \sum m_{IK}\cos\left[2\,(\omega_K + \omega_d)t + 2\theta_I\right] \qquad \text{(I-128)}$$

SPECTRUM CONSIDERATIONS

The previous section has attempted to find the output time-amplitude response of a linear detector. It would seem therefore, that due to the difficulty encountered it might be simpler to obtain, instead, the amplitude and phase signal spectrum. When we attempt to solve this problem, we are immediately confronted (for the simplest problem) with evaluating the integral

$$v_o(\omega) = \frac{1}{\pi}\int_0^{\pi} (1 + 2 R_I \cos \Delta\omega t + R_I^2)^{\frac{1}{2}} \cos (K\Delta\omega t)\, dt \qquad \text{(I-129)}$$

from equation (I-59b) and the Fourier integral theorem. This type of problem could more generally be stated as (see equation I-77)

$$v_o(\omega) = \frac{1}{\pi}\int_0^{\pi}(1 + A_1 \cos \omega_1 t + A_2 \cos \omega_2 t$$

$$+ \ldots. A_N \cos \omega_N t)^{\frac{1}{2}} \cos(K\omega_1 t)\, dt \qquad \text{(I-130)}$$

No simple solution for the first integral was found. The second cannot be represented by a Fourier series due to the incommensurable frequency terms.

The first equation can, however, be evaluated in terms of hypergeometric functions and consequently expressed with the aid of series expansions (see reference (5), equation 13.10). The details of this evaluation could also be computer automated. Further investigation is being conducted in this area.

Another approach in the spectrum area would be to solve for the power spectrum. This appears to offer certain advantages since the power spectrum (not the previously attempted amplitude and phase spectrum), due to lack of phase information, should be simpler to obtain. The direct approach would be then to obtain the power spectrum by a Fourier transform of the auto-correlation function. The auto-correlation for a deterministic signal and interference is given by

$$\phi(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} v_o(t)\, v_o\, (t-\tau)\, dt \qquad \text{(I-131)}$$

For the simplest case of equation (I-59b) this would mean evaluating

$$\phi(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} \{\left[1 + 2\ R \cos \Delta\omega t + R^2\right]\left[1 + R \cos \Delta\omega(t + \tau) + R^2\right]\}^{\frac{1}{2}}\ dt \qquad \text{(I-132)}$$

which, due to the difficulty of obtaining the average, is of no advantage over the previous time-amplitude method. However, since,

$$\left|\int f\ (x)\ dx\ \right|^2 \leq \int |f\ (x)|^2\ dx \qquad \text{(I-133)}$$

equation (I-132) can be reduced to

$$|\phi(\tau)|^2 \leq \left[1 + R^2\right]^2 + 2\ R^2 \cos \Delta\omega\tau \qquad \text{(I-134)}$$

Evaluation of the power spectrum from this approach is equal in difficulty to that of using equation (I-129). If the signal and interference process is stationary and ergodic, the auto-correlation function could also be obtained from the joint probability density of signal and interference:

$$\phi(\tau) = \text{ave}\ \{X_1 X_2 P_o(X_1,X_2)\} \qquad \text{(I-135)}$$

The basic problem with this approach is, therefore, to obtain the joint probability density of signal and interference $P_o(X_1,X_2)$. A summary of known distribution functions can be found in reference (17). Further investigation is being conducted in this area.

PHASE DETECTION

The next type of detector to be discussed is that of the phase detector. The FM detected output is the time derivative of the phase-detected signal and hence will be investigated following this discussion. In either type of detector, amplitude variations will be assumed negligible. The detectors are most commonly preceded by a limiter which fully limits the output signal for some minimum received signal. For this and all signals above this level, the amplitude output is constant and is given by

$$v_{\substack{\text{limiter}\\ \text{output}}}(t) = V_{\substack{\text{limiter}\\ \text{constant}}} \cos\left[\omega_o t + \phi(t)\right] \qquad \text{(I-136)}$$

Where

$$\phi(t) = \tan^{-1} \frac{Y(t)}{X(t)}$$

the ideal phase detector obtains

$$v_o(t) = K_\phi\{\phi(t)\} \qquad \text{(I-137)}$$

Therefore, (ignoring the detector constant) it is desired to obtain $\phi(t)$ from the previously considered interfering situations. As a first application consider the problem of AM interference on a phase modulated desired signal.

<u>Phase Detection, Application 1, a Desired Phase Modulated Signal and an Undesired AM Signal.</u> The signals of interest for a phase modulated desired signal with AM interference are given by

$$S_\phi(t) = A_S \cos\left(\omega_o t + \sum_{K=1}^{N} \Delta\phi_K \cos \omega_K t\right) \qquad \text{(I-138a)}$$

$$= \left[A_S \cos\left(\sum_{K=1}^{N} \phi_K \cos \omega_K t\right)\right] \cos \omega_o t$$

$$- \left[A_S \sin\left(\sum_{K=1}^{N} \phi_K \cos \omega_K t\right)\right] \sin \omega_o t \qquad \text{(I-138b)}$$

$$= \{A_S \cos\left[S_K(t)\right]\} \cos \omega_o t$$

$$- \{A_S \sin\left[S_K(t)\right]\} \sin \omega_o t \qquad \text{(I-138c)}$$

$$I_{AM}(t) = A_I\left(1 + \sum_{K=1}^{N} m_K \cos \omega_K t\right) \cos\left[(\omega_o + \Delta\omega)t + \theta_I\right] \qquad \text{(I-139a)}$$

$$= \{A_I\left[1 + I_K(t)\right] \cos(\Delta\omega t + \theta_I)\} \cos \omega_o t$$

$$- \{A_I \left[1 + I_K(t)\right] \sin(\Delta\omega t + \theta_I)\} \sin \omega_o t \qquad \text{(I-139b)}$$

It is, therefore, apparent that for the fully limited situation

$$v(t) = |V(t)| \cos\left[\omega_o t + \phi(t)\right] \qquad \text{(I-140)}$$

where

$|V(t)| = V$ limiter constant

$$\phi(t) = \tan^{-1}\left[\frac{A_S \sin\left[S_K(t)\right] + A_I\left[1 + I_K(t)\right]\sin\left[\Delta\omega t+\theta_I\right]}{A_S \cos\left[S_K(t)\right] + A_I\left[1 + I_K(t)\right]\cos\left[\Delta\omega t+\theta_I\right]}\right] \quad \text{(I-141a)}$$

$$= S_K(t) + \tan^{-1}\left[\frac{A_I\left[1 + I_K(t)\right]\sin\left[\Delta\omega t+\theta_I-S_K(t)\right]}{A_S+A_I\left[1+I_K(t)\right]\cos\left[\Delta\omega t+\theta_I-S_K(t)\right]}\right] \quad \text{(I-141b)}$$

Two analytical difficulties are apparent: The functions cos $\left[S_K(t)\right]$ and sin $\left[S_K(t)\right]$; and the inverse tan function. The solution to the first problem can be obtained by writing

$$f(x) = Rl\left[e^{j(k \cos x)}\right] = \cos (k \cos x) \quad \text{(I-142)}$$

and noting from reference(18), page 508 that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{j(k\cos x - n x)}\, dx = (-j)^n Jn (K) \quad \text{(I-143)}$$

The standard series solutions are given by

$$\cos (K \sin \omega_S t) = J_o (K) + 2 \sum_{n=1}^{\infty} J_{2n}(K) \cos 2 n \omega_S t \quad \text{(I-144)}$$

$$\sin (K \sin \omega_S t) = 2 \sum_{n=o} J_{2n+1}(K) \sin (2n + 1) \omega_S t \quad \text{(I-145)}$$

where

$$\left.\begin{aligned} J_o(x) &\approx \frac{\cos (x - \pi/4)}{(\pi x/2)^{\frac{1}{2}}} \\ J_1(x) &\approx \frac{\sin (x - \pi/4)}{(\pi x/2)^{\frac{1}{2}}} \end{aligned}\right\} \quad x \geq 4^*$$

The various types of Bessel functions can be found from

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*The above approximations could also be used for $x \geq 1$, with correspondingly less accurate results.

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \tag{I-146}$$

$$J_2(x) = \frac{4}{x} J_1(x) - J_o(x) \tag{I-147}$$

The expansion can therefore be written using equations (I-144) and (I-145). The answer can be shown to be

$$\cos(K\cos x) = J_o(K) - 2 J_2(K) \cos 2x + 2 J_4(K) \cos 4x - \ldots \tag{I-148}$$

$$\sin(K\cos x) = 2J_1(K) \sin x - 2 J_3(K) \sin 3x + \ldots \tag{I-149}$$

Since the periodic function given in equation (I-141) could have been written

$$\cos\left[\sum_{K=1}^{N} \phi_K \cos \omega_K t\right] = \cos\left[\phi_1 \cos \omega_1 t + \phi_2 \cos \omega_2 t + \ldots . \phi_N \cos \omega_N t\right] \tag{I-150a}$$

$$= \cos\left[\phi_2 \cos \omega_2 t + \ldots \phi_N \cos \omega_N t\right] \cos(\phi_1 \cos \omega_1 t) - \sin\left[\phi_2 \cos \omega_2 t + \ldots \phi_N \cos \omega_N t\right] \sin(\phi_1 \cos \omega_1 t) \tag{I-150b}$$

and in a similar fashion:

$$\cos\left[\sum_{K=M}^{N} \phi_M \cos \omega_M t\right] = \cos\left[\phi_M \cos \omega_M t + \ldots \phi_N \cos \omega_N t\right] \tag{I-151a}$$

$$= \cos\left[\phi_{M-1} \cos \omega_m t + \ldots + \phi_N \cos \omega_N t\right] \cos\left(\phi_M \cos \omega_M t\right) - \sin\left[\phi_{M-1} \cos \omega_M t + \ldots . + \phi_N \cos \omega_N t\right] \sin(\phi_M \cos \omega_N t) \tag{I-151b}$$

The desired answer can, therefore, be obtained from the product of series expansions of the various coefficients. The resultant answer (although the details have not been carried out due to the practical necessity of choosing values for the coefficients) is apparently still a Fourier Series of the form

$$\cos\left(\sum_{K=1}^{N} \phi_K' \, \omega_K t\right) = \phi_o' + \phi_1' \cos \omega_1 t + \phi_2' \cos \omega_2 t + \ldots \phi_N' \cos \omega_N t \qquad \text{(I-152)}$$

The numerator of equation (I-141) could be expanded in a similar form. The resulting ratio is impossible to solve other than by numerical approximation techniques. The only solutions that were found are approximations for large S/I or I/S ratios.

<u>Phase Detection, Application 2, Approximations for a Desired Phase Modulated Signal and an Undesired Tone Modulated AM Signal.</u> Consider the simpler form of equation (I-141) in which only a single pair of desired and undesired sidebands are considered. It is therefore apparent that

$$\phi(t) = \tan^{-1}\left[\frac{A_S \sin(\phi_S \cos \omega_S t) + A_I(1+m_I \cos \omega_I t)\sin(\Delta\omega t+\phi_I)}{A_S \cos(\phi_S \cos \omega_S t) + A_I(1+m_I \cos \omega_I t)\cos(\Delta\omega t+\phi_I)}\right] \qquad \text{(I-153a)}$$

$$= \phi_S \cos \omega_S t + \tan^{-1}\left[\frac{A_I(1+m_I \cos \omega_I t)\sin(\Delta\omega t+\phi_I-\phi_S)\cos \omega_S t}{A_S+A_I(1+m_I \cos \omega_I t)\cos(\Delta\omega t+\phi_I-\phi_S)\cos \omega_S t}\right] \qquad \text{(I-153b)}$$

If $A_S >> A_I$, then, with the aid of the binomial theorem

$$\frac{1}{A_S + A_I(1+m_I \cos \omega_I t)\cos(\Delta\omega t + \phi_I - \phi_S \cos \omega_S t)} \approx \frac{1}{A_S} - \frac{A_I(1+m_I \cos \omega_I t)\cos(\Delta\omega t + \phi_I - \phi_S \cos \omega_S t)}{(A_S)^{\frac{1}{2}}} + \ldots \qquad \text{(I-154)}$$

Using only the first term we obtain

$$\phi(t) = \phi_S \cos \omega_S t + \tan^{-1}\left[R_I (1+m_I \cos \omega_I t) \sin (\Delta\omega t+\phi_I-\phi_s \cos\omega_S t)\right] \tag{I-155a}$$

which for the case considered is approximately equal to

$$\phi(t) \approx \phi_S \cos \omega_S t + R_I (1 + m_I \cos \omega_I t) \sin (\Delta\omega t + \phi_I - \phi_S \cos \omega_S t) \tag{I-155b}$$

As a second approximation, consider the case when

$$A_I >> A_S$$

and from a similar type of argument (reverse the desired and undesired terms), it can be shown that

$$\phi(t) \approx \Delta\omega t + \phi_I + R_S \sin (S_K(t) - \Delta\omega t - \phi_I) \tag{I-156}$$

It is apparent that the solutions given in equations (I-155) and (I-156) are still not completely expanded. The second term in each of these equations still requires a series expansion of the type discussed previously. See equations (I-144), (I-145), (I-148) and (I-149).

<u>Phase Detection, Application 3, a Desired Phase Modulated Signal and an Undesired Multiple CW Signal.</u> As a second example consider the case of multiple interfering CW signals such that

$$I_{CW}(t) = A_{I1} \cos\left[(\Delta\omega_1 + \omega_o)t + \phi_{I1}\right] + \ldots A_{IN} \cos\left[(\Delta\omega_N + \omega_o)t + \phi_{IN}\right] \tag{I-157}$$

and consequently with the same $S_\phi(t)$ as the previous problem

$$v(t) = \left[A_S \cos (\Sigma\phi_{SK} \cos \omega_{SK} t) + A_{I1} \cos (\Delta\omega_1 t + \phi_{I1}) \ldots + A_{IN} \cos (\Delta\omega_N t + \theta_{IN})\right] \cos \omega_o t$$
$$- \left[A_S \sin (\Sigma\phi_{SK} \cos \omega_{SK} t) + A_{I1} \sin (\Delta\omega_1 t\ \theta_{I1}) \ldots + A_{IN} \sin (\Delta\omega_N t + \theta_{IN})\right] \sin \omega_o t \tag{I-158a}$$

$$= X'(t) \cos \omega_o t - Y'(t) \sin \omega_o t \tag{I-158b}$$

where we desire to obtain

$$\phi(t) = \tan^{-1} \frac{Y(t)}{X(t)} \tag{I-159a}$$

(Continued on next page)

$$= \tan^{-1}\left[\frac{A_S \sin(\Sigma\phi_{SK} \cos \omega_{SK}t) + A_{I1} \sin(\Delta\omega_1 t + \theta_{I1})}{A_S \cos(\Sigma\phi_{SK} \cos \omega_{SK}t) + A_{I1} \cos(\Delta\omega_1 t + \theta_{I1})}\right.$$ (This fraction cont'd next line)

$$\left.\frac{+\ldots A_{IN} \sin(\Delta\omega_N t + \theta_{IN})}{+\ldots A_{IN} \cos(\Delta\omega_N t + \theta_{IN})}\right]$$

$$= S_K(t) + \tan^{-1}\left[\frac{A_{I1} \sin\left[\Delta\omega_I t + \theta_{I1} - S_K(t)\right]}{A_{I1} \cos\left[\Delta\omega_I t + \theta_{I1} - S_K(t)\right]}\right.$$ (This fraction cont'd next line)

$$\left.\frac{+\ldots A_{IN} \sin\left[\Delta\omega_N t + \theta_{IN} - S_K(t)\right]}{+\ldots A_{IN} \cos\left[\Delta\omega_N t + \theta_{IN} - S_K(t)\right]}\right] \tag{I-159b}$$

where the same difficulty previously encountered occurs.

It is apparent from the previous discussions of a phase detected signal that the solutions considered are not easily solvable. The more complicated case of adding interfering phase modulated and frequency modulated signals will not be considered in detail since it is only necessary to change

$$\Delta\omega t + \theta_I - S_K(t) \rightarrow \Delta\omega t + \theta_I + I_K(t) - S_K(t)$$

$$1 + I_K(t) \rightarrow 1$$

in equation (I-141).

The answer can be simply shown by inspection to be

$$\phi(t) = S_K(t)$$

$$+ \tan^{-1}\left[\frac{A_I \sin\left[\Delta\omega t + \theta_I + I_K(t) - S_K(t)\right]}{A_S + A_I \cos\left[\Delta\omega t + \theta_I + I_K(t) - S_K(t)\right]}\right] \tag{I-160}$$

The resulting answer can be expressed again in series form with the result that the overall answer is still more complicated than equation (I-141).

<u>Phase Detection, Application 4, a Desired Phase Modulated Signal and an Undesired Pulsed Signal</u>. As the next example consider a pulsed interfering signal.

The desired phase modulated signal is given by

$$S(t) = A_S \cos\left[\omega_o t + S_K(t)\right] \tag{I-161a}$$

$$= A_S \cos\left[S_K(t)\right] \cos \omega_o t - A_S \sin\left[S_K(t)\right] \sin \omega_o t \tag{I-161b}$$

and the pulsed interfering signal is given by

$$I(t) = A_I A_{IN}(t) \cos\left[(\omega_o + \Delta\omega)t + \theta_I\right] \tag{I-162}$$

which can be rewritten as

$$I(t) = A_I A_{IN}(t) \cos\left[\Delta\omega t + \theta_I - S_K(t)\right] \cos\left[\omega_o t + S_K(t)\right] - A_I A_{IN}(t) \sin\left[\Delta\omega t + \theta_I - S_K(t)\right] \sin\left[\omega_o t + S_K(t)\right] \tag{I-163}$$

The phase detected output is therefore

$$\phi(t) = S_K(t) + \tan^{-1}\left\{\frac{A_I A_{IN}(t) \sin\left[\Delta\omega t + \theta_I - S_K(t)\right]}{A_S + A_I A_{IN}(t) \cos\left[\Delta\omega t + \theta_I - S_K(t)\right]}\right\} \tag{I-164}$$

<u>Phase Detection, Application 5, Approximations for a Desired Phase Modulated Signal and an Undesired Phase Modulated Signal.</u> As the next example consider an off-tuned phase modulated signal.

The desired signal is given by

$$S(t) = A_S \cos\left[\omega_o t + S_K(t)\right] \tag{I-165}$$

and the interfering signal is given by

$$I(t) = A_I \cos\left[(\omega_o + \Delta\omega)t + \theta_I + I_{Ko}(t)\right] \tag{I-166}$$

Rewriting this as

$$I(t) = A_I \cos\left[\Delta\omega t + \theta_I + I_{Ko}(t) - S_K(t)\right] \cos\left[\omega_o t + S_K(t)\right] - A_I \sin\left[\Delta\omega t + \theta_I + I_{Ko}(t) - S_K(t)\right] \sin\left[\omega_o t + S_K(t)\right] \tag{I-167}$$

the ideal phase detector therefore obtains

$$\phi(t) = S_K(t) + \tan^{-1}\left\{\frac{A_I \sin\left[\Delta\omega t+\theta_I+I_{Ko}(t)-S_K(t)\right]}{A_S+A_I\cos\left[\Delta\omega t+\theta_I+I_{Ko}(t)-S_K(t)\right]}\right\} \quad \text{(I-168)}$$

This is in general difficult to treat, however, for the case when

$$A_S >> A_I$$

$$\phi(t) \approx S_K(t) + R_I \sin\left[\Delta\omega t + \theta_I + I_{Ko}(t) - S_K(t)\right] \quad \text{(I-169)}$$

When $A_I >> A_S$

$$\phi(t) = \Delta\omega t + \theta_I + I_{Ko}(t) \quad \text{(I-170)}$$

<u>Phase Detection, Application 5, a Desired Phase Modulated Signal and an Undesired Single Side Band Signal</u>. Consider the problem in which the interference is an off-tuned SSB signal given by

$$I(t) = A_I \sum_{K=1}^{N} m_{IK} \cos\left[(\omega_o + \Delta\omega + \omega_{IK})t + \theta_I\right] \quad \text{(I-171a)}$$

$$\begin{aligned} I(t) = A_I\Big[&I_K(t) \cos(\Delta\omega t + \omega_{IK}t - S_K(t) + \theta_I) \\ &- I_{Ko}(t) \sin(\Delta\omega t + \omega_{IK}t - S_K(t) \\ &+ \theta_I)\Big] \cos\left[\omega_o t + S_K(t)\right] \\ &- A_I\Big[I_K(t) \sin\left[\Delta\omega t + \omega_{IK}t - S_K(t) + \theta_I\right] \\ &+ I_{Ko}(t) \cos(\Delta\omega t + \omega_{IK}t - S_K(t) \\ &+ \theta_I)\Big] \sin\left[\omega_o t + S_K(t)\right] \end{aligned} \quad \text{(I-171b)}$$

The desired signal is again the phase modulated signal given by

$$S(t) = A_S \cos\left[\omega_o t + S_K(t)\right] \quad \text{(I-172)}$$

The combined signal is found to be

$$V(t) = S(t) + I(t) = \Big\{A_S + A_I\Big[I_K(t) \cos(\Delta\omega t$$

(Continued on next page)

$$+ \omega_{IK}t - S_K(t) + \theta_I) - I_{Ko}(t) \sin(\Delta\omega t + \omega_{IK}t - S_K(t) + \theta_I)\Big] \cos\left[\omega_o t + S_K(t)\right]\Big\} \cos\left[\omega_o t + S_K(t)\right]$$

$$- \Big\{ A_I \Big[I_K(t) \sin\left(\Delta\omega t + \omega_{IK}t - S_K(t) + \theta_I\right) + I_{Ko}(t) \cos(\Delta\omega t + \omega_{IK}t - S_K(t) + \theta_I)\Big]\Big\} \sin\left[\omega_o t + S_K(t)\right] \quad \text{(I-173)}$$

The ideal phase detector obtains

$$\phi(t) = S_K(t) + \tan^{-1}\left\{\frac{A_I \sum m_{IK} \sin\left[\Delta\omega t + \omega_{IK}t - S_K(t) + \theta_I\right]}{A_S + A_I \sum m_{IK} \cos\left[\Delta\omega t + \omega_{IK}t - S_K(t) + \theta_I\right]}\right\} \quad \text{(I-174)}$$

When $A_S >> A_I$

$$\phi(t) \approx S_K(t) + R_I \sum m_{IK} \sin\left[\Delta\omega t + \omega_{IK}t - S_K(t) + \theta_I\right] \quad \text{(I-175)}$$

When $A_I >> A_S$

$$\phi(t) \approx \Delta\omega t + \sum_{K=1}^{N} \omega_{IK}t + \theta_I \quad \text{(I-176)}$$

FREQUENCY MODULATION DETECTION

The frequency modulation detector is one that ideally obtains the time derivative of the previously obtained phase detector outputs, that is,

$$v_{FM}(t) = K_{FM} \cdot \frac{1}{2\pi} \left[\frac{d\phi(t)}{dt}\right] \quad \text{(I-177)}$$

It is therefore desired to obtain $\frac{d\left[\phi(t)\right]}{dt}$. This can, in turn, be rewritten as

$$\frac{d\left[\phi(t)\right]}{dt} = \frac{d}{dt} \tan^{-1} \frac{Y(t)}{X(t)} = \frac{X^2(t)}{X^2(t) + Y^2(t)} \frac{d}{dt} \left[\frac{Y(t)}{X(t)}\right] \quad \text{(I-178)}$$

from which it is easily shown that

$$\frac{d\phi(t)}{dt} = \frac{X(t)\ \frac{dY(t)}{dt} - Y(t)\ \frac{dX(t)}{dt}}{X^2(t) + Y^2(t)} \quad \text{(I-179)}$$

The basic difference between the preceding phase detected desired signal and the present is a change in the form of the signal to

$$S_{FM}(t) = A_S \cos\ (\omega_o t + \sum_{K=1}^{N} B_K \sin \omega_{SK} t) \quad \text{(I-180)}$$

A discussion of the difference between the phase modulated and frequency modulated forms can be found in numerous references. The difference accounts for the fact that the sin ω_S t terms must be differentiated to result in the desired information terms cos ω_S t, and the difference in the series coefficients.

The preceding phase detector application 1 will now be considered.

<u>FM Detection, Application 1, a Desired FM Signal and an Undesired AM Signal</u>. From phase detector application 1 we obtain

$$Y(t) = A_S \sin\ (\sum_{K=1}^{N} B_{SK} \sin \omega_{SK} t)$$

$$+ (1 + \sum_{K=1}^{N} m_{IK} \cos \omega_{IK} t) \sin\ (\Delta\omega t + \theta_I) \quad \text{(I-181a)}$$

$$= A_S \sin \left[S_K(t)\right] + A_I \left[1 \right.$$

$$\left. + I_K(t)\right] \sin\ (\Delta\omega t + \theta_I) \quad \text{(I-181b)}$$

$$X(t) = A_S \cos\ (\sum_{K=1}^{N} B_{SK} \sin \omega_{SK} t) + A_I\ (1$$

$$+ \sum_{K=1}^{N} m_{\ IK} \cos \omega_{IK} t) \cos\ (\Delta\omega t + \theta_I) \quad \text{(I-182a)}$$

$$= A_S \cos\left[S_K(t)\right] + A_I\left[1 + I_K(t)\right] \cos\,(\Delta\omega t + \theta_I) \qquad \text{(I-182b)}$$

We therefore need to obtain

$$\frac{d\left[Y(t)\right]}{dt},\quad \frac{d\left[X(t)\right]}{dt},\quad X^2(t) \text{ and } Y^2(t).$$

These are found to be

$$\begin{aligned}\frac{d\left[Y(t)\right]}{dt} = &+ A_S \frac{d}{dt}\left[S_K(t)\right] \cos\left[S_K(t)\right] + A_I\, \Delta\omega \cos\,(\Delta\omega + \theta_I)\\ &+ A_I \sin\,(\Delta\omega t + \theta_I) \frac{d}{dt}\left[I_K(t)\right]\\ &+ A_I\, I_K(t)\, \Delta\omega \cos\,(\Delta\omega t + \theta_I)\end{aligned} \qquad \text{(I-183)}$$

$$\begin{aligned}\frac{d\left[X(t)\right]}{dt} = &-A_S \frac{d}{dt}\left[S_K(t)\right] \sin\left[S_K(t)\right]\\ &- A_I\, \Delta\omega \sin\,(\Delta\omega + \theta_I) + A_I \cos\,(\Delta\omega t + \theta_I)\, \frac{d\left[I_K(t)\right]}{dt}\\ &- A_I I_K(t)\, \Delta\omega \sin\,(\Delta\omega t + \theta_I)\end{aligned} \qquad \text{(I-184)}$$

$$\begin{aligned}X^2(t) + Y^2(t) = &\, A_S{}^2 + A_I{}^2\left[1 + I_K(t)\right]^2 + 2\, A_S\, A_I\, \Big[1\\ &+ I_K(t)\Big]\, \{\sin\,(\Delta\omega t + \theta_I) \sin\left[S_K(t)\right]\\ &+ \cos\,(\Delta\omega t + \theta_I) \cos\left[S_K(t)\right]\}\end{aligned} \qquad \text{(I-185)}$$

After rearranging terms the answer is found to be

$$\frac{d\phi(t)}{dt} = S_K{}'(t) + \frac{R_I\left[1 + I_K(t)\right]\left[\Delta\omega - S_K{}'(t)\right]\{\cos\left[\Delta\omega t + \theta_I - S_K(t)\right] + R_I\left[1 + I_K(t)\right]\}}{1 + R_I{}^2\left[1 + I_K(t)\right]^2 + 2R_I\left[1 + I_K(t)\right]\cos\left[\Delta\omega t + \theta_I - S_K(t)\right]}$$

$$+ \frac{R_I\, I_K{}'(t) \sin\left[\Delta\omega t + \theta_I - S_K(t)\right]}{1 + R_I{}^2\left[1 + I_K(t)\right]^2 + 2R_I\left[1 + I_K(t)\right]\cos\left[\Delta\omega t + \theta_I - S_K(t)\right]} \qquad \text{(I-186)}$$

where

$$S_K'(t) = \frac{d}{dt} S_K(t) = \sum_{K=1}^{N} B_K \omega_{SK} \cos \omega_{SK} t$$

$$I_K'(t) = \frac{d}{dt} I_K(t)$$

Although this answer still involves Bessel series expansions, the form is more easily handled than the previous phase detector equation (I-141), and will be consequently left in this form. Any desired approximation can easily be obtained from this expression.

FM Detection, Application 2, a Desired and Undesired FM Signal. Consider the general off-tuned frequency modulated signal

$$I_{FM}(t) = A_I \cos\left[\omega_o t + \phi_I + I_K(t)\right] \qquad \text{(I-187)}$$

The desired answer can be simply obtained from equation (I-186) by

$$\Delta\omega - S_K'(t) \rightarrow \Delta\omega + I_K'(t) - S_K'(t)$$

$$1 + I_K(t) \rightarrow 1$$

The answer is therefore found to be $\frac{d\phi(t)}{dt} = S_K'(t)$

$$+ \frac{R_I\left[\Delta\omega + I_K'(t) - S_K'(t)\right]\{\cos\left[\Delta\omega t+\phi_I+I_K(t)-S_K(t)\right] + R_I\}}{1 + R_I^2+2R_I \cos\left[\Delta\omega t+\phi_I+I_K(t)-S_K(t)\right]} \qquad \text{(I-188)}$$

This answer still requires Bessel series expansions for specific forms of the desired and undesired signals.

FM Detector, Application 3, Series Derivation of a Desired and Undesired FM Signal. The analysis in this application discusses the reformulation of equation (I-188) into a series form. Although this formulation could be obtained directly from equation (I-188), it is actually easier to begin with the phase detected output given in equation (I-160).

$$\phi(t) = S_K(t) + \tan^{-1}\left[\frac{R_I \sin\Delta}{1+R_I\cos\Delta}\right] \qquad \text{(I-189)}$$

where

$$\Delta = \Delta\omega t + \theta_I + I_K(t) - S_K(t)$$

It is apparent that equation (I-188) could be obtained by differentiating equation (I-189). It is now convenient to re-

formulate this in a series form.

Let

$$\tan \alpha = \frac{R_I \sin \Delta}{1 + R_I \cos \Delta} \qquad \text{(I-190a)}$$

and

$$\tan \alpha = \frac{K \sin \alpha}{K \cos \alpha} \qquad \text{(I-190b)}$$

and consequently

$$K = \sqrt{1 + 2R_I \cos \Delta + R_I^{\,2}} \qquad \text{(I-191)}$$

It is also true that

$$K \sin \alpha = R_I \sin \Delta \qquad \text{(I-192)}$$

$$K \cos \alpha = 1 + R_I \cos \Delta \qquad \text{(I-193)}$$

If equation (I-4) is now multiplied by $j = \sqrt{-1}$ and added to equation (I-5) we obtain

$$1 + R_I \cos \Delta + j R_I \sin \Delta = K (\cos \alpha + j \sin \alpha) \qquad \text{(I-194)}$$

This can be rewritten as

$$1 + R_I e^{j\Delta} = K e^{j\alpha} \qquad \text{(I-195)}$$

Taking the log of both sides we obtain

$$\log (1 + R_I e^{j\Delta}) = \log K + j\alpha \qquad \text{(I-196)}$$

The first half can also be expressed as the series

$$\log (1 + R_I e^{j\Delta}) = R_I e^{j\Delta} - \frac{R_I^{\,2}}{2} e^{2j\Delta} + \frac{R_I^{\,3}}{3} e^{3j\Delta} - \ldots \qquad \text{(I-197)}$$

Equating the imaginary terms of (I-9) to α we obtain the desired

result.

$$\alpha = R_I \sin \Delta - \frac{R_I^{\,2}}{2} \sin 2\Delta + \frac{R_I^{\,3}}{3} \sin 3\,\Delta - \ldots \qquad \text{(I-198a)}$$

$$\alpha = - \sum_n \frac{(-R_I)^n}{n} \sin n\Delta \qquad \text{(I-198b)}$$

The FM output is desired so that we actually want

$$\frac{d\alpha}{dt} = R_I \Delta' \cos \Delta - R_I^{\,2} \Delta' \cos 2\Delta + R_I^{\,3}\, \Delta' \cos 3\Delta - \ldots$$

$$= - \sum_{n=1}^{\infty} (-R_I)^n \cos (n\Delta)\ \frac{d\Delta}{dt} \qquad \text{(I-199)}$$

Equation (I-199) can be considered the basic FM series relationship. It is apparent, however, that the problem of differentiating Δ is not complete. The next step in the analysis is fairly long so that reference to some previous work will be made (20). In order to conform to the nomenclature of this reference, equation (I-199) will be rewritten as

$$\frac{d\alpha}{dt} = - \sum_{n=1}^{\infty} (-R_I)^n \left\{ \cos \left[n\,\psi + n(\Delta\omega t + \theta_I) \right] \right\} (\Delta\omega + \frac{d\psi}{dt}) \qquad \text{(I-200)}$$

where the simplification

$$\Delta \rightarrow \Delta\omega t + \theta_I + B_I \sin \omega_I t - B_S \sin \omega_S t \qquad \text{(I-201)}$$

has been made and ψ is defined as

$$\psi = B_I \sin \omega_I t - B_S \sin \omega_S t \qquad \text{(I-202)}$$

The terms involving ψ can in turn be expanded as

$$\cos \left[n\,\psi + n\,\Delta\omega t + n\,\theta_I \right] \left[\Delta\omega + \frac{d\psi}{dt} \right]$$

$$= \Delta\omega \cos \left[n\,(\Delta\omega t + \theta_I) \right] \cos n\,\psi$$

(Continued on next page)

$$+ \cos\left[n\,(\Delta\omega t + \theta_I)\right]\left[\cos\,(n\,\psi)\,\frac{d\,\psi}{dt}\right]$$

$$- \Delta\omega \sin\left[n\,(\Delta\omega t + \theta_I)\right] \sin n\,\psi$$

$$+ \sin\left[n\,(\Delta\omega t + \theta_I)\right]\left[\sin\,(n\,\psi)\,\frac{d\,\psi}{dt}\right] \qquad \text{(I-203)}$$

Reference (20) evaluates

$$\left[\cos\,(n\,\psi)\,\frac{d\,\psi}{dt}\right]$$

and the answer is found in equation (104) of that reference. The resulting answer for $\sin\,(n\,\psi)\,\frac{d\,\psi}{dt}$ is therefore also obtained by transforming cos to sin. It can also be noted from the derivation of equation (101) to (102) of that reference that $(\Delta\omega + \frac{d\,\psi}{dt})$. cos $(n\,\psi)$ changes the derivation of the term r/γ. This change results in

$$\frac{r}{\gamma} \rightarrow \frac{r}{\gamma} + \Delta\omega = \frac{r\omega_I}{n} + \Delta\omega \qquad \text{(I-204)}$$

With this in mind and using the basic equation given in this reference for $\begin{matrix}\cos\\ \sin\end{matrix}\,(\psi)\,\frac{d\psi}{dt}$, equation (I-5) can be rewritten as

$$\cos n\Delta\omega t\left[2\pi \sum_{r=-\infty}^{\infty}\ \sum_{s=-\infty}^{\infty}\left[\frac{rf_I}{n} - \frac{sf_S}{n}\right.\right.$$

$$\left.\left. + \Delta f\right] J_r(nB_I)\, J_S(nB_S)\, \cos\,(r\omega_I t - S\omega_S t)\right]$$ (Continued on next page)

$$- \sin n\Delta\omega t \left[2\pi \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \left[\frac{rf_I}{n} - \frac{sf_S}{n} \right.\right.$$

$$\left.\left. + \Delta f \right] J_r (n B_I) J_S (nB_S) \sin (r\omega_I t - s\omega_S t) \right] \qquad \text{(I-205)}$$

The total desired result from equation (I-200) and (I-205) is now obtained in terms of the nomenclature of this report as

$$I_o (t) = \frac{1}{2\pi} \frac{d\alpha}{dt}$$

$$= \sum_{n=1}^{\infty} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (-R_I)^n \left[\frac{rf_I}{n} - \frac{sf_S}{n} \right.$$

$$\left. + \Delta f \right] J_r (nB_I) J_S (nB_S) \cos (n\Delta\omega t + n\theta_I + r\omega_I t - s\omega_S t) \qquad \text{(I-206)}$$

The output signal has been expressed in a completely expanded series form. It is clear that expanding this for the cases of more sideband terms would become increasingly involved. At this point it is convenient to re-express equation (I-199) in an equivalent form. It should be noted that

$$\cos (n\Delta) \cdot \frac{d\Delta}{dt} = \frac{1}{n} \frac{d}{dt} \sin n\Delta \qquad \text{(I-207)}$$

Equation (I-199) therefore becomes

$$\frac{d\alpha}{dt} = - \frac{d}{dt} \sum_{n=1}^{\infty} \frac{(-R_I)^n}{n} \sin n\Delta \equiv - \frac{d}{dt} \left\{ f(t) \right\} \qquad \text{(I-208)}$$

It is evident that if f(t) has a Fourier transform f(ω), i.e.,

$$f(t) \longleftrightarrow F(\omega) \qquad \text{(I-209)}$$

then by a tranform property (reference[6], equation 6.201) we find that

$$\frac{d}{dt}\left\{f\ (t)\right\} \longleftrightarrow \omega f(\omega) \qquad \text{(I-210)}$$

The resulting form of equation (I-208) is therefore more easily visualized (but not computed) than the form of (I-206).

SYNCHRONOUS DETECTORS

A synchronous detector is one that ideally produces the product of a desired signal,

$$s(t) = S_K(t) \cos\ (\omega_o t + \theta_S) \qquad \text{(I-211)}$$

with a synchronous carrier

$$\cos\ (\omega_o t + \theta_r) \qquad \text{(I-212)}$$

to produce the product

$$v(t) = \frac{S_K(t)}{2}\left[\cos\ (\theta_S - \theta_r)\right.$$

$$\left. + \cos\ (2\ \omega_o t + \theta_S + \theta_r)\right] \qquad \text{(I-213)}$$

The carrier term is then filtered to obtain the desired output

$$v_o(t) = K_{syn}\frac{S_K(t)}{2}\cos\ (\theta_S - \theta_r) \qquad \text{(I-214)}$$

when,

$$\theta_S = \theta_r$$

and

$$K_{syn} = 1$$

$$v_o(t) = \frac{S_K(t)}{2} \qquad \text{(I-215)}$$

Although it is desired to keep the two signals exactly in phase, a phase difference may exist due to random or deterministic causes. The noise or random case will be discussed in the next section. In the other case, it is convenient to choose a reference such that $\theta_r=0$,

and leave θ_S arbitrary (but constant).

For this case, the desired output signal can be written

$$v_o(t) = \frac{S_K(t)}{2} \cos \theta_S \tag{I-216}$$

where,

$$0 \leq |V_o(t)| \leq \frac{S_K(t)}{2}$$

It is, therefore, paramount to keep $\theta_S \approx 0^\circ$.

Consider the general input signal consisting of a desired signal and interference which can be written

$$v(t) = X(t) \cos \omega_o t - Y(t) \sin \omega_o t \tag{I-217a}$$

$$= \left[X^2(t) + Y^2(t)\right]^{1/2} \cos \left[\omega_o t + \phi(t)\right] \tag{I-217b}$$

The detected output is obtained as

$$v_o(t) = \frac{\left[X^2(t) + Y^2(t)\right]^{1/2}}{2} \cos \left[\phi(t)\right] \tag{I-218}$$

However, since

$$\phi(t) = \tan^{-1} \frac{Y(t)}{X(t)} = \cos^{-1} \frac{X(t)}{\left[X^2(t) + Y^2(t)\right]^{1/2}} \tag{I-219}$$

the output is simply found to be

$$v_o(t) = \frac{X(t)}{2} \tag{I-220}$$

This simple result is due to the fact that synchronous detectors are linear detectors, since multiplication is a linear property. The synchronous detector will now be discussed for various types of interference.

<u>Synchronous Detector, Application 1, a Desired DSBSC Signal and an Undesired AM Signal</u>. Consider a desired double sideband suppressed carrier signal

$$S_{DSB}(t) = A_S\ S_K(t) \cos (\omega_o t + \theta_S) \tag{I-221a}$$

$$= \left[A_S\ S_K(t) \cos \theta_S\right] \cos \omega_o t - \left[A_S\ S_K(t) \sin \theta_S\right] \sin \omega_o t \tag{I-221b}$$

and the AM interfering signal

$$I_{AM}(t) = A_I \left[1 + I_K(t)\right] \cos\left[(\omega_o + \Delta\omega) t + \theta_I\right] \tag{I-222}$$

Combining the desired signal and interference we obtain

$$v(t) = \left[A_S\ S_K(t) \cos \theta_S + A_I \{1+I_K(t)\}\cos(\Delta\omega t+\theta_I)\right] \cos \omega_o t \tag{I-223a}$$

$$-\left[A_S S_K(t) \sin \theta_S + A_I\{1+I_K(t)\}\sin(\Delta\omega t+\theta_I)\right] \sin \omega_o t$$

$$= X(t) \cos \omega_o t - Y(t) \sin \omega_o t \tag{I-223b}$$

The output signal is, therefore, simply

$$v_o(t) = \frac{A_S}{2}\ S_K(t) \cos \theta_S$$

$$+ \frac{A_I}{2} \left[1 + I_K(t)\right] \cos (\Delta\omega t + \theta_I) \tag{I-224a}$$

$$= V_{oSSB}(t) + \frac{X_I}{2} (t) \tag{I-224b}$$

Synchronous Detector, Application 2, a Desired SSB Signal and and Undesired AM Signal. Assume a single sideband signal with a phase difference for the desired signal of θ_S

$$S_{SSB}(t) = A_S \sum_{K=1}^{n} m_{SK} \cos \left[(\omega_{SK} + \omega_o)t + \theta_S\right] \tag{I-225a}$$

$$= \left\{\left(A_S \sum_{K=1}^{N} m_{SK} \cos \omega_{SK} t\right) \cos \theta_S - \left(A_S \sum_{K=1}^{N} m_{SK} \cos \omega_{SK} t\right) \sin \theta_S\right\} \cos \omega_o t$$

(Continued on next page)

$$- \left\{ \left(A_S \sum_{K=1}^{N} m_{SK} \cos \omega_{SK} t \right) \cos \theta_S + \left(A_S \sum_{K=1}^{N} m_{SK} \cos \omega_{SK} t \right) \sin \theta_S \right\} \sin \omega_o t \qquad \text{(I-225b)}$$

For an interfering signal given by

$$I_{AM} = A_I \left[1 + I_K(t) \right] \cos \left[(\omega_o + \Delta\omega)t + \theta_I \right] \qquad \text{(I-226)}$$

the synchronous detected output is given by

$$V_o(t) = \frac{X(t)}{2}$$

$$= \left(\frac{A_S}{2} \sum_{K=1}^{N} m_{SK} \cos \omega_{SK} t \right) \cos \theta_S$$

$$- \left(\frac{A_S}{2} \sum_{K=1}^{N} m_{SK} \cos \omega_{SK} t \right) \sin \theta_S$$

$$+ \frac{A_I}{2} \left[1 + I_K(t) \right] \cos \left(\Delta\omega t + \theta_I \right) \qquad \text{(I-227)}$$

<u>Synchronous Detector, Application 3, a Desired Unsuppressed Carrier SSB Signal and an Undesired Signal</u>. As the next example consider the SSB case of an unsuppressed carrier (UC) where the desired signal is given by

$$S_{SSB-UC}(t) = A_S \cos (\omega_o t + \theta_S) + S_{SSB}(t) \qquad \text{(I-228)}$$

Since this only adds a constant to the x(t) function, the output is given by

$$V_o(t) = \frac{A_S}{2} + V_{oSSB}(t) \qquad \text{(I-229)}$$

<u>Synchronous Detector, Application 4, a Desired Synchronous Signal and an Undesired Pulsed Signal</u>. Consider the example of pulse interference given by

$$I(t) = A_I \, A_{IN}(t) \cos \left[(\omega_o + \Delta\omega)t + \theta_I \right] \qquad \text{(I-230)}$$

The synchronous detector interference output is found to be

$$X_I(t) = \frac{A_I}{2} A_{IN}(t) \cos(\Delta\omega t + \theta_I) \tag{I-231}$$

<u>Synchronous Detector, Application 5, a Desired Synchronous Signal and an Undesired AM Signal</u>. As the next example consider the interfering AM tone given by

$$I(t) = A_I (1 + m_I \cos \omega_I t) \cos(\Delta\omega t + \theta_I) \cos \omega_o t$$

$$- A_I (1 + m_I \cos \omega_I t) \sin(\Delta\omega t + \theta_I) \sin \omega_o t \tag{I-232}$$

The synchronous detector output is found to be

$$\frac{X_I(t)}{2} = \frac{A_I}{2} (1+m_I \cos \omega_I t) \cos(\Delta\omega t + \theta_I) \tag{I-233}$$

The general AM interference is therefore found to be

$$\frac{X_I(t)}{2} = \frac{A_I}{2} \left[1 + I_K(t)\right] \cos(\Delta\omega t + \theta_I) \tag{I-234}$$

APPENDIX II

DETECTION MODELING OF RANDOM NOISE FOR A LARGE CARRIER TO NOISE CONDITION

Appendix I discussed the point that for a large signal-to-noise ratio, the signal and noise can approximately be treated independently of each other and in the limiting cases the noise can be neglected in computing the signal-to-interference ratio. In certain cases it is desired not to neglect the noise but to take into account a first order correction factor. This appendix presents the necessary expressions for the output noise power. The following is devoted to a discussion of the output noise power for large carrier-to-noise ratios.

The narrowband input noise to the detector can be described by the narrowband quadrature form

$$N(t) = X_N(t) \cos \omega_o t - Y_N(t) \sin \omega_o t \qquad \text{(II-1)}$$

where,

$$Y_N(t) = \sum_{m=1}^{M} \left[2\, G_H(\omega_m)\, \Delta f\right]^{\frac{1}{2}} \sin\left[(\omega_m - \omega_o)t + \theta_m\right] \qquad \text{(II-2)}$$

$$X_N(t) = \sum_{m=1}^{M} \left[2\, G_H\, (\omega_m)\, \Delta f\right]^{\frac{1}{2}} \cos\left[(\omega_m - \omega_o)t + \theta_m\right] \qquad \text{(II-3)}$$

and

$G_H(\omega_m)$ = power spectral density. The quantities $X_N(t)$ and $Y_N(t)$ are independent random variables of a slowly varying frequency compared to f_o. In order to emphasize the previous method of analysis it is necessary to write equation (II-1) with additional signal carrier given by

$$S_{CW}(t) = A_S \cos \omega_o t \qquad \text{(II-4)}$$

The composite signal can now be written in the narrowband form as

$$v(t) = S_{CW}(t) + N(t) \qquad \text{(II-5a)}$$

$$= \left[A_S + X_N(t)\right] \cos \omega_o t - \left[Y_N(t)\right] \sin \omega_o t \qquad \text{(II-5b)}$$

$$= \left[A_S^2 + 2\, A_S\, X_N(t) + X_N^2 + Y_N^2(t)\right]^{1/2} \cos \left[\omega_o t + \phi(t)\right] \qquad \text{(II-5c)}$$

where

$$\phi(t) = \tan^{-1} \left[\frac{Y_N(t)}{A_S + X_N(t)}\right] \qquad \text{(II-6)}$$

Since $X_N(t)$, $Y_N(t)$ and $\phi(t)$ are random variables, it is desired to find the average detected outputs.

The first problem to be investigated is that of the linear detector. The average output can be obtained from reference (2) as the envelope density function.

$$g(r) = \frac{r}{N} \exp \left[-r^2 + A_S^2/2N\right] \; I_o \left(\frac{r\, A_S}{N}\right) \qquad \text{(II-7)}$$

It is desired to obtain

$$\overline{|V(t)|} = \int_o^\infty r\, g(r)\, dr \qquad \text{(II-8)}$$

when

$\overline{|V(t)|}$ symbolizes the average linear detector output

The general moments of equation (II-8) have been given by reference (2) equations (3.10-12). The average output can be obtained from equation (II-8) as

$$|V(t)| = A\left[1 + \frac{N}{2A^2} + \frac{9N^2}{8A^4} + \ldots\right] \qquad \text{(II-9)}$$

For the case of a large carrier, only the first two terms need be used and this can in turn be rewritten as*

$$\overline{|V(t)|} = A + \frac{\overline{X_N^2(t)} + \overline{Y_N^2(t)}}{4A} \qquad \text{(II-10a)}$$

---

*This could also be obtained by using the first two terms of the binomial expansion.

$$= A + N/A \tag{II-10b}$$

where

$$\overline{Y_N{}^2} = \overline{X_N{}^2} = \sum_{m=1}^{M} G_H(\omega_m)\ \Delta f \tag{II-11}$$

The total output noise power is

$$N = \sum_{m=1}^{M} G_H(\omega_m)\ \Delta f \tag{II-12}$$

The quantity N is the total mean noise power in a one-ohm resistor. The average power (the first moment squared) is, therefore, approximately

$$\overline{|V(t)|}^2 = A_S{}^2 + \frac{\left[\overline{X_N{}^2} + \overline{Y_N{}^2}\right]}{2} + \frac{\left[\overline{X_N{}^2} + \overline{Y_N{}^2}\right]^2}{16\ A_S{}^2} \tag{II-13a}$$

$$= A_S{}^2 + N + N^2/4\ A_S{}^2 \tag{II-13b}$$

The average output of a square law detector is found directly from equation (II- 5) to be

$$\overline{|V(t)|^2} = A_S{}^2 + \overline{X_N{}^2} + \overline{Y_N{}^2} + 2\ A_S\ \overline{X_N(t)} \tag{II-14}$$

If, as for the usual case of receiver and atmospheric noise,

$\overline{X_N(t)} = 0$ this reduces to

$$\overline{|V(t)|^2} = A_S{}^2 + \overline{X_N{}^2} + \overline{Y_N{}^2} \tag{II-15a}$$

$$= A_S{}^2 + 2N \tag{II-15b}$$

If interference considerations make it necessary to assume a gaussian distributed interference whose average value is not equal to zero, equation (II-14) must be used.

The average phase and instantaneous frequency will next be

discussed. The average phase is usually zero. However, the average square phase is actually desired (since power is a more useful criteria).

For the case of a large carrier-to-noise ratio, equation (II-6) can be rewritten as

$$\phi(t) \approx \tan^{-1}\left[\frac{Y_N(t)}{A_S}\right] \quad \text{(II-16a)}$$

$$\approx \frac{Y_N(t)}{A_S} \quad \text{(II-16b)}$$

when

$$A_S >> \overline{Y_N(t)}$$

From equation (II-3) this can also be written as

$$\phi(t) = \frac{1}{A_S} \sum_{m=1}^{M} \left[2\, G_H(\omega_m)\ \Delta f\right]^{\frac{1}{2}} \sin\,(\omega_m t + \theta_m) \quad \text{(II-17)}$$

The average phase for a frequency ω_m is obviously

$$\overline{\phi\omega_m\ (t)} = 0 \quad \text{(II-18)}$$

The average phase detected power at a frequency ω_m and a phase detector constant of unity, is

$$\overline{\phi^2\ \omega_m(t)} = \frac{1}{A_S{}^2} \sum_{m=1}^{M} \overline{\left[2\, G_H(\omega_m)\ \Delta f\right] \sin^2\,(\omega_m t + \theta_m)} \quad \text{(II-19a)}$$

$$= \frac{\overline{Y_N{}^2(t)}}{A_S{}^2} \quad \text{(II-19b)}$$

The frequency modulation detector obtains the derivative of equation

(II-17) and for a unity detector obtains

$$d\left[\phi(t)\right] = \frac{1}{A_S} \sum_{m=1}^{M} \frac{d}{dt} (\omega_m t + \theta_m)\left[2\ G_H\ (\omega_m)\Delta f\right]^{\frac{1}{2}} \cos\ (\omega_m t + \theta_m)$$

$$= \frac{1}{A_S} \sum_{m=1}^{M} \omega_m\left[2G_H\ (\omega_m)\Delta f\ \right]^{\frac{1}{2}} \cos\ (\omega_m t + \theta_m) \qquad \text{(II-20a)}$$

$$+ \frac{1}{A_S} \sum_{m=1}^{M} \frac{d}{dt}\ \theta_m\left[2\ G_H\ (\omega_m)\ \Delta f\right]^{\frac{1}{2}} \cos\ (\omega_m t + \theta_m) \qquad \text{(II-20b)}$$

Since the random variable, θ_m, is approximately a constant and $\frac{d\ \theta_m}{dt} \simeq 0$. The average frequency modulation **for** a frequency, ω_m, is, therefore

$$\frac{d}{dt}\ \left[\overline{\phi\omega_m\ (t)}\right] = 0 \qquad \text{(II-21)}$$

The normalized average frequency modulated power for a frequency ω is

$$\overline{\left[\frac{d}{dt}\ \left[\phi\omega_m\ (t)\right]\right]^2} = \frac{{\omega_m}^2}{{A_S}^2}\ \overline{{Y_N}^2} \qquad \text{(II-22)}$$

The synchronous detector produces the noise product

$$v(t) = A_S \cos\ \omega_o t\ .\ N(t) \qquad \text{(II-23a)}$$

$$= A_S\ X_N(t) \cos^2\ \omega_o t\ -\ A_S Y_N(t) \cos\ \omega_o t\ \sin\ \omega_o t \qquad \text{(II-23b)}$$

$$= A_S\ X_N(t)\left[\tfrac{1}{2} + \tfrac{1}{2} \cos\ 2\ \omega_o t\right] - A_S Y_N(t)\left[\tfrac{1}{2} \sin\ 2\omega_o t\right] \qquad \text{(II-23c)}$$

$$= \frac{A_S\ X_N(t)}{2} + \frac{A_S}{2}\left[X_N(t)\cos 2\ \omega_o t - Y_N(t)\sin 2\ \omega_o t\right] \qquad \text{(II-23d)}$$

Since the synchronous detector is followed by a low-pass filter the output signal is

$$v_o(t) = \frac{A_S\ X_N(t)}{2} \qquad \text{(II-24)}$$

This again shows that the output noise power can be treated independently but without the restriction of a large carrier-to-noise ratio.

The detector outputs obtained from equations (II-10), (II-14), (II-19) and (II-24) are the low-pass detector outputs. In order to obtain the filtered output power for a particular low-pass filter, $H_{LP}(\omega)$, it is still necessary to obtain

$$P_o(\omega) = \int_{o}^{\omega_{B\omega/2}} |H_{LP}(\omega)|^2\ S_P(\omega)\ d\omega \qquad \text{(II-25)}$$

where the proper detector outputs $\left[S_P\ (\omega)\right]$ have been substituted and it is assumed that the maximum frequency of integration does not approach the carrier frequency.

The general output of a nonlinear device to a noise input containing a number of frequency components is all possible cross product combinations of the input frequencies, equation (II-9). For the special case of a large carrier, the previous discussion has shown that the detected noise components (power) do not produce cross product terms and can, therefore, be treated as independent of each other.

APPENDIX III

IF OFF-TUNING SIGNAL MODIFICATIONS

The basic problem to be discussed in this appendix is the effect or modification that the IF amplifier produces on the off-tuned interfering signal. For most interfering cases except that of SSB, the input signals are even; symetrical functions. The off-tuning displaces these even, symetrical signals to an unsymetrical location in the IF filter characteristic. The resulting output signal is, therefore, also an unsymetrical function. This process is symbolized in Figure (III-1). Although the output signal for a general IF characteristic is completely unsymetrical, certain IF characteristics transform the even input signal to odd, but not unsymetrical, signals. That is, functions of the form

$$H_I(\omega) = K\,\omega^{-n_I} \qquad \text{(III-1)}$$

perform the desired operation. Since any filter characteristic can be constructed from

$$H(\omega) = \sum_I H_I(\omega) \qquad \text{(III-2)}$$

this representation is completely general.

The object of this appendix is, however, to use a single power law representation and to obtain a modification of the input signal of a form similar to the original signal. Only two off-tuned pro-blems basically need to be considered. These are the problems of an AM and an FM undesired signal to a desired AM signal. The FM and phase detector problem need not be considered, since the linear detector models used for these cases are only valid within the constant amplitude filter characteristic region. The only remaining variation for this problem is that due to the linear phase characteristics, which only results in a constant delay being added to the interfering signal. As an important example of an AM desired signal, consider an off-tuned AM interference signal. The input signal is given by

$$I'(t) = A_I\,(1 + M_I \cos \omega_I t) \cos \left[(\omega_o + \Delta\omega)t + \theta_I\right] \qquad \text{(III-3)}$$

Let the filter characteristic be given by equation (III-1). The filter output signal is then given by

$$I(t) = \frac{KA_I}{\Delta\omega^n} \cos \left[(\omega_o + \Delta\omega)t + \theta_I\right] + \frac{K}{(\Delta\omega+\omega_I)^n}\frac{(A_I m_I)}{2} \cos \left[(\omega_o + \Delta\omega + \omega_I)t + \theta_I\right]$$

(Continued on next page)

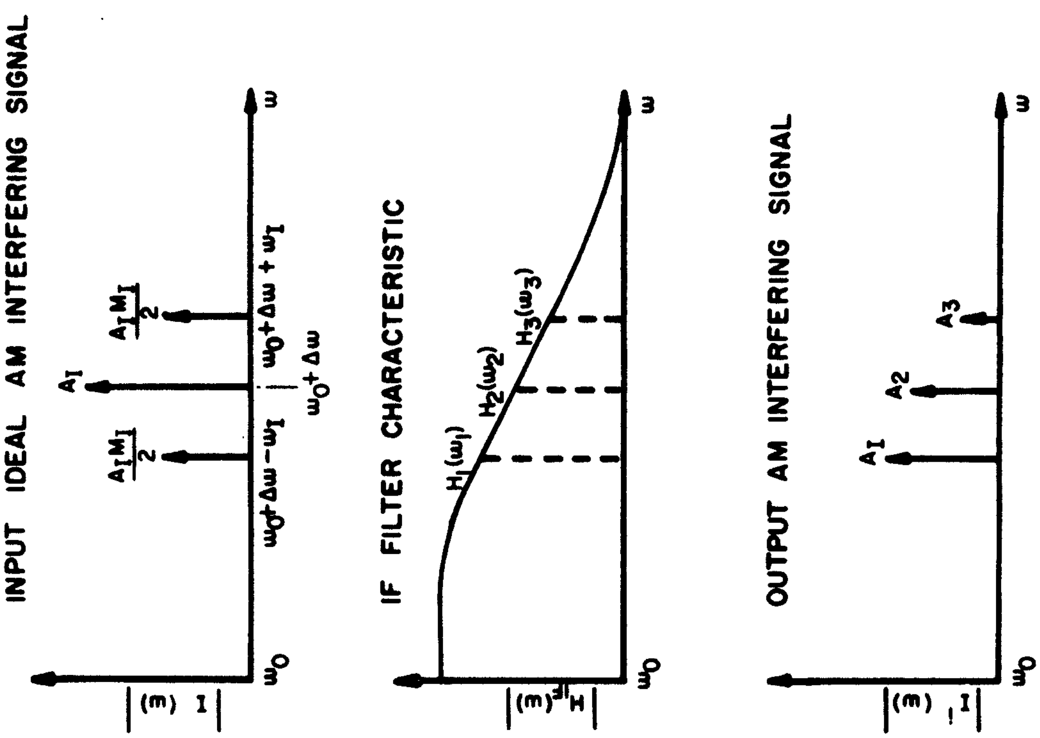

Figure III-1. IF Transformation of AM Interference

$$+ \frac{K}{(\Delta\omega-\omega_I)^n} \frac{A_I m_I}{2} \cos\left[(\omega_o + \Delta\omega - \omega_I)t+\theta_I\right] \qquad \text{(III-4)}$$

which, after some trignometric manipulation, can be rewritten as

$$I(t) = \frac{KA_I}{\Delta\omega^n} \left[1 + m_I^2 \cos^2 \omega_I t\right]^{\frac{1}{2}} \cos\left[(\omega_o+\Delta\omega)t+\theta_I(t)\right]$$

where

$$\theta_I(t) = \left\{\tan^{-1}\left[\frac{m_I \cos \omega_I t}{1+ m_I \sin \omega_I t}\right] + \theta_I\right\} \qquad \text{(III-5)}$$

When $m_I^2 << 1$ this can be approximated by

$$I(t) \approx \frac{KA_I}{\Delta\omega^n} \left[1 + m_I^2 \cos^2\omega_I t\right] \cos \left[(\omega_o+\Delta\omega)t+ \theta_I(t)\right] \qquad \text{(III-6a)}$$

$$\approx \frac{KA_I}{\Delta\omega^n} \left[1 + \frac{m_I^2}{2} \cos 2\omega_I t\right] \cos \left[(\omega_o+\Delta\omega)t + \theta_I(t)\right] \qquad \text{(III-6b)}$$

where $m_I^2 << 1$ and has subsequently been neglected.

This is, then, of the same form as the equation (II-3) and the relations derived within APPENDIX II can be used providing the proper amplitude and frequency terms from equation (III-6) are used

As a second example, consider the off-tuned FM signal

$$I'(t) = A_I \cos \left[(\omega_o+\Delta\omega)t + \theta_I + B_I \sin \omega_I t\right]$$

$$= A_I \left[\cos (B_I \sin \omega_I t) \cos (\omega_o + \Delta\omega t + \theta_I)\right.$$

$$\left. - \sin (B_I \sin \omega_I t) \sin (\omega_o + \Delta\omega t + \theta_I)\right] \qquad \text{(III-7)}$$

The instantaneous frequency is given by

$$f_{inst} = \frac{1}{2\pi} \frac{d}{dt} \left[\phi(t)\right] = \omega_o + \Delta\omega + B_I\omega_I \cos \omega_I t \qquad \text{(III-8)}$$

The frequency is, therefore, varying sinusoidally about $\omega_o + \Delta\omega$. For an amplitude characteristic given by equation (II-1), this results in the amplitude variation

$$A_I(t) = \left[\frac{A_o}{(\Delta\omega + f_I B_I \cos \omega_I t)^n}\right] \qquad \text{(III-9)}$$

For the special filter conditions or those when $\Delta\omega >> f_I B_I$

$$A_I(t) \approx A_o \; \Delta\omega^{-n} + \left(\frac{n \; f_I B_I}{\Delta\omega^{\;n-1}}\right) \cos \omega_I t \qquad \text{(III-10)}$$

The result is that, for these assumptions, the FM signal is converted into an effective AM signal given by

$$I'(t) = A_I \left[A_o \; \Delta\omega^{-n} - \left(\frac{n \; f_I B_I}{\Delta\omega^{\;n-1}}\right) \cos \omega_I t\right] \cos\,(\omega_o + \Delta\omega)t \qquad \text{(III-11)}$$

This, again, shows that the off-tuning of the IF can be accounted for by taking into account the attenuation of the carrier and modifying the sideband terms to fit that of ordinary AM modulation as given by equations (III-6) or (III-11).

DISCUSSION OF RESULTS

The analysis shown in this appendix is an abbreviated discussion of many possible examples that could be chosen. It was given to show that under certain special restrictions the basic forms of the undesired signals into the second detector are those of the general input signal categories of AM, FM, SSB, etc. This basic approach eliminates the analysis of complex modulated signals that would change the analysis to an untractable form.

REFERENCES

1. L.J. Greenstein, D. Fryberger, B. Ebstein, Pulse Interference Analysis of Communication Receivers, ECAC Technical Note MD-123, May 1964.

2. S.O. Rice, "Mathematical Analysis of Random Noise", BSTJ Vols. 23 and 24.

3. Rand Corporation, Table of Q Function, 1 January 1950.

4. G.N. Coffey, A.B. Glenn, Performance Analysis of Three Types of FSK Detection Techniques, RCA Technical Report 58-581-2.

5. D. Middleton, Introduction to Statistical Communication Theory, McGraw-Hill, 1960.

6. S.J. Mason, H.J. Zimmerman, Electronic Circuits Signals and Systems, John Wiley and Sons, 1960.

7. R. Mayher, "Off-Tuning Effects Produced by Interference", Ninth Tri-Service Conference on Electromagnetic Compatibility, October 1963.

8. H.S. Black, Modulation Theory, D. Van Nostrand Co., 1953.

9. W.B. Davenport, Jr., An Introduction to the Theory of Random Signals and Noise, McGraw-Hill, 1958.

10. P. Christopher, R.J. Mayher, Probability of Threshold Detecting Off-Tuned Pulses, ECAC Technical Memo MD-25, January 25, 1963.

11. R. Mayher, Interference Detection Modeling, TM X003-18, June 1963.

12. N. Wilde, A.J. Hoehn, L.A. Follett, "A Digital Representation of Interference within a Communication Receiver Including the Demodulation Process", Seventh Conference on Electromagnetic Compatibility, November 1961.

13. C.B. Aiken, "Theory of the Detection of Two Modulated Waves by a Linear Rectifier", IRE, April 1933.

14. E. Jahnke, F. Emde, Tables of Functions, Dover Publications, 194

15. Bierens D. Haan, Nouvelles Tables, Hafner Publishing Co., 1957.

16. W.B. Jones, Method of Calculating the Effects of AM Interference on AM Systems, AD 2/4492, March 1959.

17. B. Haller, A Summary of Known Distribution Functions, Rand-T-27, 7 January 1953.

18. E.A. Guillemin, The Mathematics of Circuit Analysis, The Technology Press, 1949.

19. R. Mayher, J. Pierzga, I. Pressman, Degradation as a Function of Interference Modulation Type, ECAC Section Technical Note MD-84, October 1963.

20. M.S. Corrington, "Frequency Modulation Distortion Caused by Common and Adjacent-Channel Interference", RCA Review, December 1946.

21. G.E. Renand, The Effects of Interference on Speech Communications, RADC-TN-104, June 1960.

UNCLASSIFIED

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DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Electromagnetic Compatibility Analysis Center, USNMEL, Annapolis, Maryland	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

INTRODUCTION TO THE ANALYSIS OF INTERFERENCE TO THE PERFORMANCE OF COMMUNICATIONS SYSTEMS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

5. AUTHOR(S) *(Last name, first name, initial)*

Mayher, R.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1965	239	21

8a. CONTRACT OR GRANT NO. AF 19(628)-4154 b. PROJECT NO. 649E c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) ECAC-TR-65-1 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* ESD-TR-65-16

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